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# Quantum Information Theory – Sheet 3

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**Webpage:** *http://www.thp.uni-koeln.de/∼rk/qit_22.html/*

**Submission** of solutions as pdf-file until Thursday, May 19, 12 pm, to
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## 9. Random vectors of $s^\perp \subset \mathbb{Z}_2^n$ *3+3+3 Punkte*

In the end, Simon's algorithm produces a set of random vectors $l_1, \ldots, l_K$ of the $(n-1)$ - dimensional linear subspace $V = s^\perp$ of $\mathbb{Z}_2^n$. Here we want to convince ourself that the vectors $l_1, \ldots l_K$ span with high probability the entire subspace $V$, provided that $K$ is significantly larger than $n$ but still of order $O(n)$. To this end show the following:

**a)** If $l_1, \ldots, l_k \in V$ span a subspace $W \subsetneq V$, adding another random vector $l_{k+1}$ of $V$, the vectors $l_1, \ldots, l_k, l_{k+1}$ span with probability $p \geq \frac{1}{2}$ a subspace $W'$ of increased dimension $\dim W' = \dim W + 1$.

**b)** Assuming that the probability in **a)** would be exactly $p = \frac{1}{2}$, show that $K$ random vectors of $V$ span a subspace of dimension $d < n-1$ with probability

$$\frac{1}{2^K} \binom{K}{d} .$$

**c)** Argue by the result of **b)** that for $K = 4n$ $(\gg 1)$ random vectors $l_1, \ldots, l_K$ of $V$ fail to span the entire space $V$ with a probability much less than $e^{-n/2}$.

**Hint:** approximate the binomial distribution by a normal distribution

## 10. Quantum Fourier Transformation *4+4 Punkte*

The discrete Fourier transformation over $\mathbb{Z}_N$ with $N = 2^n$ can be implemented as a unitary transformation $F_n$ on $n$ qubits, given by

$$F_n \left| x \right\rangle := \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{\frac{2\pi i}{N} xk} \left| k \right\rangle .$$

**a)** Show by an explicit calculation that $F_n$ is unitary.

**b)** Design a quantum circuit for the *inverse* discrete Fourier transformation.

## 11. Fourier Transformation of an $r$-periodic function *2+6+2+2 Punkte*

Let a function $f\,:\,\mathbb{Z}_N \to \mathbb{C}$ be given by

$$f(x) = \sqrt{\frac{r}{N}} \times \begin{cases} 1 & : \quad x + l = 0 \mod r \\ \\ 0 & : \quad x + l \neq 0 \mod r \end{cases} \qquad ,$$

where $N$ is a positive integer, $r$ a positive integer that divides $N$, and $l$ a positver integer less than $r$.

**a)** Draw the graph of $f$ for $N = 24$, $r = 6$ and $l = 2$.

**b)** Compute the discrete Fourier transform

$$\hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x)\, \mathrm{e}^{\frac{2\pi i}{N} xk}$$

of $f$.

**c)** Draw the graph of $|\hat{f}|$ for $N = 24$, $r = 6$ and $l = 2$.

**d)** What are the Fourier transforms of the constant function $f(x) = 1/\sqrt{N}$ and the alternating function $g$ with $g(x) = \sqrt{\frac{2}{N}}$ for even $x$ and $g(x) = 0$ for odd $x$ ?