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Quantum Information Theory – Sheet 5

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Webpage: *http://www.thp.uni-koeln.de/∼rk/qit_22.html/*

Submission of solutions as pdf-file until Thursday, June 30, 12 pm, to
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16. Commutator *4 points*

The Solovay-Kitaev theorem for efficient unitary approximation uses following relation for rotations on the Bloch-sphere:

$$\mathcal{R}_{3,\varepsilon}^{-1}\,\mathcal{R}_{1,\varepsilon}^{-1}\,\mathcal{R}_{3,\varepsilon}\,\mathcal{R}_{1,\varepsilon} \;=\; \mathcal{R}_{2,-\varepsilon^2} \;\; + \mathcal{O}(\varepsilon^3),$$

where $R_{l,\alpha} = \mathrm{e}^{i\alpha\sigma_l}$. Verify this relation and give a geometrical interpretation of it.

17. Depolarizing channel *2+2+2=6 points*

a) A quantum operation $\mathcal{E}$ on a qubit is given by

$$\mathcal{E}(\rho) = \frac{1}{4}\,(\rho + X\rho X + Y\rho Y + Z\rho Z\,)\,.$$

Show that

$$\mathcal{E}(I) = I, \quad \text{and} \quad \mathcal{E}(X) = \mathcal{E}(Y) = \mathcal{E}(Z) = 0\,.$$

b) Use the Bloch-vector representation of a general qubit state ρ in order to show that the operation $\mathcal{E}$ defined in **a)** satisfies $\mathcal{E}(\rho) = \frac{1}{2}I$.

c) The *depolarizing qubit channel* is described by the map

$$\mathcal{E}_p(\rho) = (1-p)\rho \;+\; p\,\frac{I}{2}, \tag{1}$$

where $p \in [0,1]$. Show that $\mathcal{E}_p$ can be written in the form

$$\mathcal{E}_p(\rho) \;=\; \left(1 - \frac{3}{4}p\right)\rho \;+\; \frac{p}{4}\,(X\rho X + Y\rho Y + Z\rho Z)\;.$$

18. Kraus-operator expansion *3+3=6 points*

a) Give Kraus-operator expansions for the following two quantum operations on a quantum system with Hilbert space $\mathcal{H}$ of dimension d:

$$\mathcal{E}_1(\rho) = |\psi\rangle\langle\psi|, \quad (|\psi\rangle \in \mathcal{H}), \qquad \mathcal{E}_2(\rho) = \frac{1}{d}\,I\,.$$

b) Quantum operations $\mathcal{E}$ and $\mathcal{F}$ on a quantum system Q are given by Kraus operators $E_1, \ldots, E_K$ and $F_1, \ldots F_L$, respectively. Find Kraus-operator expansions for the quantum operations $\mathcal{E} \circ \mathcal{F}$ and $\mathcal{E} \otimes \mathcal{F}$.
[The operation $\mathcal{E} \otimes \mathcal{F}$ on $Q \otimes Q$ is defined by $\mathcal{E} \otimes \mathcal{F}(\rho \otimes \sigma) = \mathcal{E}(\rho) \otimes \mathcal{F}(\sigma)$, and by linearity for general density operators.]

19. Number of Kraus-operators *10 points*

The Kraus-operators $E_1, \ldots E_K$ of a quantum operation $\mathcal{E}$ on a quantum system Q can be transformed to an equivalent set of Kraus-operators $F_1, \ldots F_K$ by a unitary matrix $U = (u_{lk})$,

$$F_l = \sum_k u_{lk} E_k \ .$$

Use this fact in order to show that one can expand $\mathcal{E}$ with at most d^2 Kraus-operators, where d is the dimension of the Hilbert space $\mathcal{H}_Q$.

[Hints: Show that the matrix $W := (\mathrm{Tr}\, E_i^+ E_j)_{ij}$ is hermitian and of rank $\leq d^2$, and thus can be unitarily transformed to a diagonal matrix $\tilde{W} = UWU^+$ with at most d^2 non-vanishing eigenvalues. Use U to transform from $\{E_k\}$ to operators $\{F_l\}$ satisfying $\mathrm{Tr}\, F_l^+ F_m = \tilde{W}_{lm}$.]