Lösungshinweise Blatt 12

49)
$$S^2 = \left(\frac{\hbar}{2}\vec{\sigma}\right)^2 = \frac{\hbar^2}{4}(\sigma_1^2+\sigma_2^2+\sigma_3^2) = \frac{3}{4}\hbar^2 \mathbb{1}_2$$
$$\sigma_i^2 = \mathbb{1}$$

$$S_+ = \frac{\hbar}{2}(\sigma_1 + i\sigma_2) = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad S_- = (S_+)^\dagger = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

$$\rightsquigarrow \quad [S^2, S_e] = \frac{3}{8}\hbar^2[\mathbb{1}, \sigma_e] = 0$$

$$[S_3, S_+] = \frac{\hbar^2}{2}\left[\begin{pmatrix}1&\\&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right] = \hbar^2\begin{pmatrix}0&1\\0&0\end{pmatrix} = \hbar S_+$$
$$[S_3, S_-] = \frac{\hbar^2}{2}\left[\begin{pmatrix}1&\\&-1\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right] = \hbar^2\begin{pmatrix}0&0\\1&0\end{pmatrix} = \hbar S_-$$

$$S_+|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\0\end{pmatrix}$$

$$S_-|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} = \hbar|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}0\\0\end{pmatrix}.$$

50a)
$$[J_e, J_m] = [L_e + S_e, L_m + S_m]$$

$$= [L_e, L_m]\otimes\mathbb{1}_S + \mathbb{1}_B\otimes[S_e, S_m]$$

$$[L_e, S_m] = [L_e\otimes\mathbb{1}_S, \mathbb{1}_B\otimes S_m] = 0$$

$$= i\hbar\,\varepsilon_{emk}(L_k\otimes\mathbb{1}_S + \mathbb{1}_B\otimes S_k) = i\,\varepsilon_{emk} J_k.$$

50 b)

$$J_3 |u\rangle = L_3 |u\rangle + S_3 |u\rangle = (\underbrace{L_3 |l,l\rangle}_{\hbar l |l,l\rangle}) \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + |l,l\rangle \otimes (\underbrace{S_3 |\tfrac{1}{2},\tfrac{1}{2}\rangle}_{\hbar \frac{1}{2} |\frac{1}{2},\frac{1}{2}\rangle})$$

$$= \hbar (l + 1/2) |u\rangle = \hbar j |u\rangle \quad \text{mit} \quad j = l + 1/2 \; ;$$

$$J_+ |u\rangle = L_+ |u\rangle + S_+ |u\rangle = (\underbrace{L_+ |l,l\rangle}_{=0}) |\tfrac{1}{2},\tfrac{1}{2}\rangle + |l,l\rangle \otimes (\underbrace{S_+ |\tfrac{1}{2},\tfrac{1}{2}\rangle}_{=0})$$

$$= 0 \; ;$$

$$J_- J_+ = (J_1 - i J_2)(J_1 + i J_2) = J_1^2 + J_2^2 + i \underbrace{[J_1, J_2]}_{i\hbar J_3} = J_1^2 + J_2^2 - \hbar J_3$$

$$\rightarrow \vec{J}^2 = J_- J_+ + J_3^2 + \hbar J_3$$

und

$$\vec{J}^2 |u\rangle = J_- \underbrace{J_+ |u\rangle}_{=0} + (J_3^2 + \hbar J_3) |u\rangle = \hbar^2 j(j+1) |u\rangle$$

$$L^2 |u\rangle = L^2 \otimes \mathbb{1}_S |u\rangle = (L^2 |l,l\rangle) \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar^2 l(l+1) |u\rangle,$$

$$S^2 |u\rangle = \mathbb{1}_B \otimes S^2 |u\rangle = |l,l\rangle \otimes S^2 |\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar^2 \tfrac{3}{4} |u\rangle \; ,$$

D.h. $|u\rangle \equiv |j, j\rangle$ = Drehimpulseigenzustand

von $\vec{J}$ der Quantenzahlen $j = l + 1/2$, $m = j$

$$\rightarrow |j, m\rangle = c_m (J_-)^{j-m} |j, j\rangle$$

für $m = -j, -j+1, \ldots, j-1, j$

c)

$$J_3 |n\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{J_3 |l,l\rangle|\tfrac12,-\tfrac12\rangle} - \underbrace{J_3 |l,l-1\rangle|\tfrac12,+\tfrac12\rangle}\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\hbar(l-\tfrac12)|l,l\rangle|\tfrac12,-\tfrac12\rangle - \hbar(l-\tfrac12)|l,l-1\rangle|\tfrac12,\tfrac12\rangle\Big)$$

$$= \hbar(l-1/2)|n\rangle = \hbar j |n\rangle \quad \text{mit } j = l - 1/2$$

$$J_+|n\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{(L_+ + S_+)|l,l\rangle|\tfrac12,-\tfrac12\rangle}_{=\,|l,l\rangle|1/2,+1/2\rangle} - \underbrace{(L_+ + S_+)|l,l-1\rangle|\tfrac12,\tfrac12\rangle}_{=\,|l,l\rangle|1/2,1/2\rangle}\Big)$$

$$= 0 \;!$$

$$\rightsquigarrow \quad J^2|n\rangle = (J_- J_+ + J_3^2 + \hbar J_3)|n\rangle$$

$$= \hbar^2 j(j+1)|n\rangle \quad ; \quad j = l - 1/2$$

$$L^2|n\rangle = \hbar^2 l(l+1)|n\rangle,$$

$$S^2|n\rangle = \hbar^2 \tfrac34 |n\rangle,$$

$$L_3|n\rangle = \frac{\hbar}{\sqrt{2}}\Big(l\,|l,l\rangle|\tfrac12,-\tfrac12\rangle - (l-1)|l,l-1\rangle|\tfrac12,\tfrac12\rangle\Big) \not\parallel |n\rangle$$

d.h. $|n\rangle$ kein EZ von L_3;

ebenso $|n\rangle$ kein EZ von S_3.

$\rightarrow \quad |n\rangle \equiv |j, m\rangle$ Drehimpulseigenzustand

von $\vec{J}$ der Quantenzahlen $j = \ell - 1/2$, $m = j$!

51) radiale S. Gl.: $\quad E\, u(r) = -\dfrac{\hbar^2}{2m} u''(r) + \dfrac{\hbar^2 \ell(\ell+1)}{2m\, r^2} \qquad (*)$

mit $r \in [R, R+a]$ und Randbedingungen

$u(R) = u(R+a) = 0$;

Wegen $a \ll R$ und $r \in [R, R+a]$

$$\frac{1}{r^2} = \frac{1}{R^2} + \sigma(a/R) \approx 1/R^2 \qquad (**)$$

$\rightarrow$ in guter Näherung $(*)$ äquivalent zu

$$\underbrace{\left(E - \frac{\hbar^2 \ell(\ell+1)}{2m\, R^2}\right)}_{= \tilde{E}_\ell} u(r) = -\frac{\hbar^2}{2m} u''(r)$$

$\hat{=}$ Teilchen im Kasten der Breite a

$$\rightarrow \tilde{E}_{\ell,n} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot n^2 \quad \rightarrow \quad E_{\ell,n} = \frac{\hbar^2}{2m}\left(\left(\frac{\pi n}{a}\right)^2 + \frac{\ell(\ell+1)}{R^2}\right)$$

$2\ell+1$ fach entartet; (a/R - Korrekturen aufgrund Näh. $(**)$ müsste man mit Störungstheorie best. ....)

52 a) $E_n(\lambda) = \hbar\omega\,(n+\tfrac{1}{2}) + \lambda\,\frac{\hbar\omega}{\ell}\,\overbrace{\langle n|x|n\rangle}^{=0}$

$$+\lambda^2(\hbar\omega)^2\sum_{m\neq n}\frac{|\langle m|\,x/\ell\,|n\rangle|^2}{E_n - E_m}$$

mit $x/\ell = (a^\dagger + a)/\sqrt{2}$ folgt

$$\langle m|\,x/\ell\,|n\rangle = \frac{1}{\sqrt{2}}\underbrace{\langle m|a^\dagger|n\rangle}_{=\delta_{n+1,m}\sqrt{n+1}} + \frac{1}{\sqrt{2}}\underbrace{\langle m|a|n\rangle}_{=\delta_{n-1,m}\sqrt{n}}$$

$$\rightarrow |\langle m|\,x/\ell\,|n\rangle|^2 = \frac{n+1}{2}\,\delta_{n+1,m} + \frac{n}{2}\,\delta_{n-1,m}$$

$$\rightarrow \hbar\omega\sum_{m\neq n}\frac{|\langle m|x/\ell|n\rangle|^2}{\underbrace{\hbar\omega(n-m)}_{=E_n-E_m}} = \frac{1}{2}\left(\frac{n+1}{-1} + \frac{n}{+1}\right) = -\frac{1}{2} \quad \rightarrow$$

d. h. $E_n(\lambda) = \hbar\omega\,(n+\tfrac{1}{2}) - \lambda^2\,\frac{\hbar\omega}{2}$

50 b) $\frac{m\omega^2}{2} x^2 + \lambda \hbar\omega \frac{x}{\ell}$

$$= \frac{\hbar\omega}{2}\Big(\underbrace{\frac{m\omega}{\hbar}}_{= 1/\ell^2} x^2 + 2\lambda \frac{x}{\ell} \Big)$$

$$= \frac{\hbar\omega}{2\ell^2}\left(x^2 + 2\lambda \ell x \right) = \frac{\hbar\omega}{2\ell^2}\left((x+\lambda\ell)^2 - \lambda^2\ell \right)$$

$$= \frac{m\omega^2}{2}(x+\lambda\ell)^2 - \lambda^2 \frac{\hbar\omega}{2}$$

d. h. $$H(\lambda) = \underbrace{p^2/2m + \frac{m\omega^2}{2} \underbrace{(x+\lambda\ell)}_{= x'}{}^2}_{\Rightarrow\ \hbar\omega(n+1/2)} \underbrace{- \lambda^2 \frac{\hbar\omega}{2}}_{\Delta E(\lambda)}$$