# Lösungshinweise Blatt 2

7) Rechnungen in Komponenten bzgl. ONB $(\varphi_{z+}, \varphi_{z-})$:

a)

$$|\langle \varphi_{x+}, \varphi_{y+}\rangle|^2 = \tfrac{1}{4}\left|\left\langle \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\i\end{pmatrix}\right\rangle\right|^2 = \tfrac{1}{4}\left(1+|i|^2\right) = 1/2$$

$$|\langle \varphi_{z+}, \varphi_{y-}\rangle|^2 = \tfrac{1}{2}\left|\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\i\end{pmatrix}\right\rangle\right|^2 = 1/2$$

b)

$$\psi = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1+i\end{pmatrix} \;\rightarrow\; \|\psi\|^2 = \frac{1}{6}\left(4+|1+i|^2\right) = 1$$

c) $+\mu_B$ gemessen mit Wkt.

$$p_+ = |\langle \varphi_{y+}, \psi\rangle|^2 = \frac{1}{12}\left|\left\langle \begin{pmatrix}1\\i\end{pmatrix}, \begin{pmatrix}2\\1+i\end{pmatrix}\right\rangle\right|^2 = \frac{1}{12}\left|2+i+1\right|^2 = \frac{5}{6}$$

$-\mu_B$ gemessen mit Wt $\quad p_- = 1 - p_+ = 1/6$

$$\rightarrow \quad \langle \mu_y\rangle_\psi = \mu_B\left(5/6 - 1/6\right) = \frac{2}{3}\mu_B .$$

8 a) Bzgl. ONB $B = (\varphi_{z+}, \varphi_{z-})$:

$$\left.\begin{aligned} P_{x+}\varphi_{z+} &= \langle\varphi_{x+},\varphi_{z+}\rangle\varphi_{x+} = \tfrac{1}{\sqrt{2}}\varphi_{x+} = \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \\ P_{x+}\varphi_{z-} &= \langle\varphi_{x+},\varphi_{z-}\rangle\varphi_{x+} = \tfrac{1}{\sqrt{2}}\varphi_{x+} = \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}\right\} \rightarrow P_{x+} = \tfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$\left.\begin{aligned} P_{x-}\varphi_{z+} &= \langle\varphi_{x-},\varphi_{z+}\rangle\varphi_{x-} = \tfrac{1}{\sqrt{2}}\varphi_{x-} = \tfrac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} \\ P_{x-}\varphi_{z-} &= \langle\varphi_{x-},\varphi_{z-}\rangle\varphi_{x-} = -\tfrac{1}{\sqrt{2}}\varphi_{x-} = \tfrac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\end{aligned}\right\} \rightarrow P_{y-} = \tfrac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

$$\rightarrow \quad \frac{\hat{\mu}_x}{\mu_B} = P_{x+} - P_{x-} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = \sigma_1$$

$$\left.\begin{aligned} P_{y+}\varphi_{z+} &= \langle\varphi_{y+},\varphi_{z+}\rangle\varphi_{y+} = \tfrac{1}{\sqrt{2}}\varphi_{y+} = \tfrac{1}{2}\begin{pmatrix}1\\i\end{pmatrix} \\ P_{y+}\varphi_{z-} &= \langle\varphi_{y+},\varphi_{z-}\rangle\varphi_{y+} = -\tfrac{i}{\sqrt{2}}\varphi_{y+} = \tfrac{1}{2}\begin{pmatrix}-i\\1\end{pmatrix}\end{aligned}\right\} \rightarrow P_{y+} = \tfrac{1}{2}\begin{pmatrix}1&-i\\i&1\end{pmatrix}$$

$$\left.\begin{aligned} P_{y-}\varphi_{z+} &= \langle\varphi_{y-},\varphi_{z+}\rangle\varphi_{y-} = \tfrac{1}{\sqrt{2}}\varphi_{y-} = \tfrac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix} \\ P_{y-}\varphi_{z-} &= \langle\varphi_{y-},\varphi_{z-}\rangle\varphi_{y-} = \tfrac{i}{\sqrt{2}}\varphi_{y-} = \tfrac{1}{2}\begin{pmatrix}i\\1\end{pmatrix}\end{aligned}\right\} \rightarrow P_{y-} = \tfrac{1}{2}\begin{pmatrix}1&i\\-i&1\end{pmatrix}$$

$$\rightarrow \quad \frac{\hat{\mu}_y}{\mu_B} = P_{y+} - P_{y-} = \begin{pmatrix}0&-i\\i&0\end{pmatrix} = \sigma_2$$

$$\left.\begin{aligned} &\left.\begin{aligned} P_{z+}\varphi_{z+} &= \varphi_{z+} = \begin{pmatrix}1\\0\end{pmatrix} \\ P_{z+}\varphi_{z-} &= 0 = \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}\right\} \rightarrow P_{z+} = \begin{pmatrix}1&0\\0&0\end{pmatrix} \\ &\left.\begin{aligned} P_{z-}\varphi_{z+} &= 0 = \begin{pmatrix}0\\0\end{pmatrix} \\ P_{z-}\varphi_{z-} &= \varphi_{z-} = \begin{pmatrix}0\\1\end{pmatrix}\end{aligned}\right\} \rightarrow P_{z-} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\end{aligned}\right\} \rightarrow \frac{\hat{\mu}_z}{\mu_B} = P_{z+} - P_{z-} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \sigma_3$$

8a)

$P_{x+}$ besitzt EWe $1, 0$ zu EVen $\varphi_{x+}, \varphi_{x-}$

$P_{x-}$ " " $0, 1$ zu EVen $\varphi_{x+}, \varphi_{x-}$

$\rightarrow \sigma_1 = P_{x+} - P_{x}$ besitzt EWe $1, -1$ zu EVen

$$\varphi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \varphi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

analog: $\sigma_2 = P_{y+} - P_{y-}$ besitzt EWe $1, -1$ zu EVen $\varphi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, $\varphi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ besitzt offenbar EWe $1, -1$ zu EVen $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$.

9a) $\hat{\mu}_z = \mu_B \, |z+\rangle\langle z-| - \mu_B \, |z-\rangle\langle z-|$

b) $\langle \mu_z \rangle_\psi = \langle \psi | \hat{\mu}_z | \psi \rangle$

$$= \frac{\mu_B}{5}\Big( \langle z+| - 2i\langle z-| \Big)\Big( |z+\rangle\langle z+| - |z-\rangle\langle z-| \Big)\Big( |z+\rangle + 2i\, |z-\rangle \Big)$$

$$= \frac{\mu_B}{5}\big(1 - 4\big) = -\frac{3}{5}\mu_B .$$

⌈ alternativ mit 8a):

$$\langle \mu_z \rangle_\psi = \langle \psi, \hat{\mu}_z \psi \rangle = \mu_B \langle \psi, \sigma_3 \psi \rangle = \frac{\mu_B}{5}\left\langle \begin{pmatrix}1\\2i\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\2i\end{pmatrix}\right\rangle$$

$$= \frac{\mu_B}{5}\left\langle \begin{pmatrix}1\\2i\end{pmatrix}, \begin{pmatrix}1\\-2i\end{pmatrix}\right\rangle = -\frac{3}{5}\mu_B . \quad ⌋$$

c) $(|x+\rangle, |x-\rangle)$ ONB von $\mathcal{H}$ und somit $\mathbb{1}_{\mathcal{H}} = |x+\rangle\langle x+| + |x-\rangle\langle x-|$

$$\rightarrow \hat{\mu}_z = \mathbb{1}_{\mathcal{H}} \hat{\mu}_z \mathbb{1}_{\mathcal{H}} = \mu_B\big( |x+\rangle\langle x+| + |x-\rangle\langle x-| \big)\big( |z+\rangle\langle z+| - |z-\rangle\langle z-| \big)\big( |x+\rangle\langle x+| + |x-\rangle\langle x-| \big)$$

$$= \frac{\mu_B}{2}\Big( \cancel{|x+\rangle\langle x+|} + |x+\rangle\langle x-| - \cancel{|x+\rangle\langle x+|} + |x+\rangle\langle x-|$$

$$+ |x-\rangle\langle x+| + \cancel{|x-\rangle\langle x-|} + |x-\rangle\langle x+| - \cancel{|x-\rangle\langle x-|} \Big)$$

$\langle x+ | z\pm \rangle = 1/\sqrt{2}$

$\langle x- | z\pm \rangle = \pm 1/\sqrt{2}$

$$= \mu_B \big( |x+\rangle\langle x-| + |x-\rangle\langle x+| \big) .$$

⌈ $\rightarrow$ bzgl. $\tilde{B} = (|x+\rangle, |x-\rangle)$: $\left(\hat{\mu}_z\right)_{\tilde{B}}^{\tilde{B}} = \mu_B \begin{pmatrix}0&1\\1&0\end{pmatrix} = \mu_B \sigma_1$ . ⌋

10)

$$\xrightarrow{\psi_0} \boxed{\tilde{M}_1} \xrightarrow{\psi_1} \boxed{\tilde{M}_2} \xrightarrow{\psi_2} \cdots \longrightarrow \boxed{\tilde{M}_k} \xrightarrow{\psi_k} \boxed{\tilde{M}_{k+1}} \xrightarrow{\psi_{k+1}}$$

Induktion über $k$:

$k = 1$ ✓

$k \to k+1$: von den Messungen $M_1, \dots, M_k$ nach I. V. höchstens eine Messung positiv;

1. Fall: für $1 \le i \le k$ Messung $M_i$ positiv mit Wkt. $\tilde{p}_i = |\langle \varphi_i, \psi_0 \rangle|^2 \rightarrow \psi_k = \varphi_i$ und Messung $M_{k+1}$ positiv mit Wkt $p_{k+1} = |\langle \varphi_{k+1}, \varphi_i \rangle|^2$ $= 0$ wegen $\varphi_{k+1} \perp \varphi_i$; d.h. $M_{k+1}$ negativ mit Wkt. $1 \rightarrow \psi_{k+1} = \bar{P}_{\varphi_{k+1}} \varphi_i = \varphi_i$ da $\varphi_i \in \varphi_{k+1}^{\perp}$.

D.h. von den Messungen $M_1, \dots, M_k, M_{k+1}$ ist $M_i$ positiv mit Wkt $p_i = \tilde{p}_i \cdot 1 = \tilde{p}_i = |\langle \varphi_i, \psi_0 \rangle|^2$ und $\psi_{k+1} = \varphi_i$

2. Fall: Messungen $M_1, \dots, M_k$ negativ, dann nach I. V.

$$\psi_k = \frac{1}{\sqrt{q_k}} \bar{P}_k \psi_0 \quad \text{und } M_{k+1} \text{ positiv mit}$$

$$\text{Wkt. } \tilde{p}_{k+1} = |\langle \varphi_{k+1}, \psi_k \rangle|^2 = \frac{1}{q_k} |\langle \varphi_{k+1}, \bar{P}_k \psi_0 \rangle|^2$$

$$= \frac{1}{q_k} |\langle \bar{P}_k \varphi_{k+1}, \psi_0 \rangle|^2 \underset{\varphi_{k+1} \in (\operatorname{span}\{\varphi_1, \dots, \varphi_k\})^{\perp}}{=} \frac{1}{q_k} |\langle \varphi_{k+1}, \psi_0 \rangle|^2 \quad (*)$$

Wkt., dass von $M_1, \dots, M_k, M_{k+1}$ genau $M_{k+1}$ positiv damit gegeben durch

$$p_{k+1} = \underbrace{q_k}_{\text{Wkt., dass } M_1, \dots, M_k \text{ negativ}} \cdot \underbrace{\tilde{p}_{k+1}}_{\text{Wkt. dass } M_{k+1} \text{ positiv}} \overset{(*)}{=} |\langle \varphi_{k+1}, \psi_0 \rangle|^2 ;$$

dazu $\psi_{k+1} = \varphi_{k+1}$

$M_{k+1}$ negativ mit Wkt. $1-\tilde{p}_{k+1} = \frac{1}{q_k}\left(q_k - |\langle\varphi_{k+1},\psi\rangle|^2\right)$

$\rightarrow$ $M_1, \ldots, M_k, M_{k+1}$ negativ mit Wkt

$$q_{k+1} = q_k\,(1-\tilde{p}_{k+1}) = q_k - |\langle\varphi_{k+1},\psi\rangle|^2 = 1 - \sum_{i=1}^{k+1} |\langle\varphi_i,\psi\rangle|^2 .$$

In diesem Fall

$$\psi_{k+1} = \frac{1}{\sqrt{1-\tilde{p}_{k+1}}}\,\bar{P}_{\varphi_{k+1}}\,\psi_k$$

$$= \frac{1}{\sqrt{1-\tilde{p}_{k+1}}}\,\frac{1}{\sqrt{q_k}}\,\bar{P}_{\varphi_{k+1}}\,\bar{P}_k\,\psi_0$$

Proj. auf $(\mathrm{Span}\{\varphi_1,\ldots,\varphi_k\})^\perp$

Projek. auf $\varphi_{k+1}^\perp$

$$= \frac{1}{\sqrt{q_{k+1}}}\,\bar{P}_{k+1}\,\psi_0$$

$\bar{P}_{\varphi_{k+1}}\,\bar{P}_k = \bar{P}_{k+1}$ da $\varphi_{k+1} \perp \mathrm{Span}\{\varphi_1,\ldots,\varphi_k\}$

$\blacksquare$