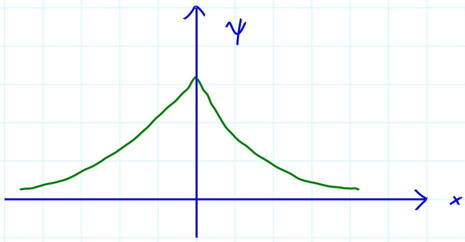
Lösungshinweise Blatt 4

a)



b)

$$\|\psi\|^2 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = \frac{1}{a}\int_{-\infty}^{\infty} dx\; e^{-2|x|/a}$$

$$= \frac{2}{a}\int_0^{\infty} dx\; e^{-2x/a} = \frac{2}{a}\left(-\frac{a}{2}\, e^{-2x/a}\right)\Big|_0^{\infty} = 1$$

c)

$$\langle x\rangle_\psi = \langle\psi|\hat{x}|\psi\rangle = \frac{1}{a}\int_{-\infty}^{\infty} dx\; x\, e^{-2|x|/a} = 0$$

$$\langle x^2\rangle_\psi = \langle\psi|\hat{x}^2|\psi\rangle = \frac{1}{a}\int_{-\infty}^{\infty} dx\; x^2 e^{-2|x|/a} = \frac{2}{a}\int_0^{\infty} dx\; x^2 e^{-2x/a}$$

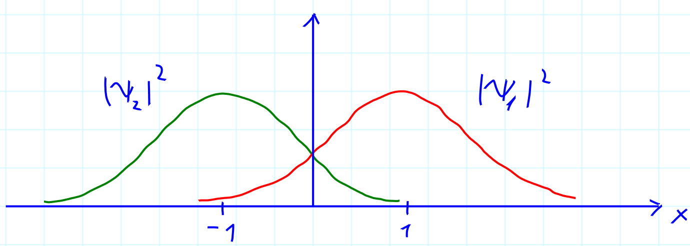
$$\underset{x=\frac{a}{2}y}{=} \frac{a^2}{4}\int_0^{\infty} dy\; y^2 e^{-y} = a^2/2$$

d)

$$P(x>a) = \int_a^{\infty} dx\, |\psi(x)|^2 = \frac{1}{a}\int_a^{\infty} e^{-2x/a} = \frac{1}{a}\left(-\frac{a}{2}\, e^{-2x/a}\right)\Big|_a^{\infty}$$

$$= \frac{1}{2e^2} = 0{,}068\ldots$$

17 a)



b)

$|\psi_{1/2}|^2$ symmetrisch um $\pm a$ $\rightsquigarrow$ $\langle x\rangle_{\psi_{1/2}} = \pm a$

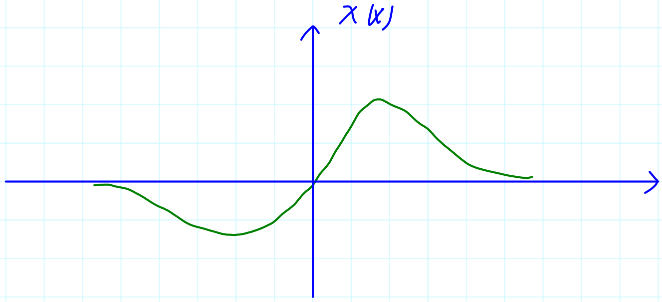
c)

$$\langle\psi_1|\psi_2\rangle = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dx\; e^{-\frac{x^2}{2\sigma^2}}}_{=1}\; e^{-a^2/2\sigma^2} = e^{-a^2/2\sigma^2}.$$

d) $1 \overset{!}{=} \langle \chi | \chi \rangle = \alpha^2 \left(\langle \psi_1 | - \langle \psi_2 | \right) \left(| \psi_1 \rangle + | \psi_2 \rangle \right)$

$$= \alpha^2 \big(\underbrace{\| \psi_1 \|^2}_{=1} + \underbrace{\| \psi_2 \|^2}_{=1} - \underbrace{\langle \psi_2 | \psi_1 \rangle}_{= e^{-a^2/2\sigma^2}} - \underbrace{\langle \psi_1 | \psi_2 \rangle}_{= e^{-a^2/2\sigma^2}} \big)$$

$$= 2\alpha^2 \left(1 + e^{-a^2/2\sigma^2} \right) \rightsquigarrow \alpha = \left(2 + 2 e^{-a^2/2\sigma^2} \right)^{-1/2}$$



$\langle x \rangle_\chi = 0$ wegen Symmetrie von $|\chi(x)|^2$ bzgl $x = 0$

20 a) $B[A,C] + [A,B]C = \cancel{BAC} - BCA + ABC - \cancel{BAC} = [A, BC]$

b) Induktion $n = 0$ ✓

$n \to n+1 \qquad [A, B^{n+1}] = [A, B B^n] \overset{a)}{=} B[A, B^n] + [A,B] B^n$

$$\overset{\text{I.V}}{=} B\, n\, c\, B^{n-1} + c\, B^n = c\,(n+1)\, B^n \qquad \blacksquare$$

c) $f(x) = \sum_{n=0}^{\infty} a_n x^n \rightsquigarrow [A, f(B)] = \sum_{n=0}^{\infty} a_n [A, B^n]$

$$\overset{b)}{=} c \sum_{n=1}^{\infty} a_n\, n\, B^{n-1} = c\, f'(B)$$

d) $[\hat{x}, \hat{p}] = i\hbar \quad , \quad [\hat{p}, x] = -i\hbar$

$\to [\hat{x}, \hat{p}^2] = 2i\hbar \hat{p} \quad , \quad [\hat{p}, \hat{x}^3] = -3i\hbar \hat{x}^2$

$[\hat{p}, e^{-a\hat{x}^2}] = +2i\hbar a \hat{x}\, e^{-a\hat{x}^2}$

$[\hat{x}, e^{-\frac{i}{\hbar} s \hat{p}}] = s\, e^{-is\hat{p}/\hbar}$

$[\hat{p}, e^{ik\hat{x}}] = \hbar k\, e^{ik\hat{x}}$

21)

a) $$\frac{d}{ds}|\psi_s\rangle = \underbrace{\frac{d}{ds}T(s)}_{= -\frac{i}{\hbar}\hat{p}\,T(s)}|\psi(0)\rangle = -\frac{i}{\hbar}\hat{p}\,|\psi_s\rangle \qquad (*)$$

$$\rightarrow \frac{d}{ds}\langle A\rangle_{\psi_s} = \frac{d}{ds}\langle \psi_s, A\,\psi_s\rangle = \langle \psi_s', A\,\psi_s\rangle + \langle \psi_s, A\,\psi_s'\rangle$$

$$\overset{(*)}{=} \langle -\frac{i}{\hbar}\hat{p}\,\psi_s, A\,\psi_s\rangle - \langle \psi_s, \frac{i}{\hbar}A\,\hat{p}\,\psi_s\rangle$$

$$= \frac{i}{\hbar}\langle \psi_s, \hat{p}A\,\psi_s\rangle - \frac{i}{\hbar}\langle \psi_s, A\hat{p}\,\psi_s\rangle = \langle \frac{i}{\hbar}[\hat{p}, A]\rangle_{\psi_s}$$

b) „$\Rightarrow$"

A translat. invariant

$$\Rightarrow \forall \psi_0 \quad 0 = \frac{d}{ds}\langle A\rangle_{\psi_s}\Big|_{s=0} \overset{a)}{=} \langle \frac{i}{\hbar}[\hat{p}, A]\rangle_{\psi_0}$$

$$\Rightarrow [\hat{p}, A] = 0 \qquad \checkmark$$

„$\Leftarrow$" $$[\hat{p}, A] = 0 \Rightarrow \frac{d}{ds}\langle A\rangle_{\psi_s} \overset{a)}{=} \langle \frac{i}{\hbar}\underbrace{[\hat{p}, A]}_{=0}\rangle_{\psi_s} = 0 \qquad \checkmark$$

c) $$\frac{d}{ds}\langle \hat{x}\rangle_{\psi_s} \overset{a)}{=} \langle \frac{i}{\hbar}\underbrace{[\hat{p}, \hat{x}]}_{= -i\hbar}\rangle_{\psi_s} = 1$$

$$\rightarrow \langle \hat{x}\rangle_{\psi_s} = \langle \hat{x}\rangle_{\psi_0} + s \,.$$