Lösungshinweise Blatt 5

23) $n=0$: $\int_0^\infty e^{-x}\,dx = 1 = 0!$ ✓

$n \to n+1$: $\int_0^\infty x^{n+1} e^{-x}\,dx = -x^{n+1}e^{-x}\Big|_0^\infty + (n+1)\int_0^\infty x^n e^{-x}\,dx \overset{\text{I.V.}}{=} (n+1)!$

24a) $\| |\psi_0\rangle \|^2 = \frac{1}{\pi a_0^3}\, 4\pi \int_0^\infty dr\, r^2 e^{-2r/a_0} \underset{r=a_0x/2}{=} \frac{1}{2}\int_0^\infty dx\, x^2 e^{-x} \overset{23)}{=} 1\,.$

b) $\langle \vec{x}\rangle_{\psi_0} = \vec{0}$ und $\langle \vec{p}\rangle_{\psi_0} = \vec{0}$ aufgrund Symmetrie;

$$\langle |\vec{x}|^2\rangle_{\psi_0} = \frac{1}{\pi a_0^3}\, 4\pi \int_0^\infty dr\, r^4 e^{-2r/a_0} \underset{r=a_0x/2}{=} \frac{a_0^2}{8}\int_0^\infty dx\, x^4 e^{-x} \underset{23)}{=} 3a_0^2\,.$$

$$\langle |\vec{p}|^2\rangle_{\psi_0} = \frac{1}{\pi a_0^3}\, 4\pi \int_0^\infty dr\, r^2 e^{-r/a_0}\left(-\hbar^2 \Delta e^{-r/a_0}\right);$$

mit $\Delta e^{-r/a_0} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} e^{-r/a_0}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-\frac{r^2}{a_0} e^{-r/a_0}\right)$

$$= \frac{1}{a_0 r^2}\left(-2r + \frac{r^2}{a_0}\right) = -\frac{2}{a_0 r} + \frac{1}{a_0^2}$$

folgt: $\langle |\vec{p}|^2\rangle_{\psi_0} = \frac{8\hbar^2}{a_0^4}\int_0^\infty dr\, r\, e^{-2r/a_0} - \frac{4\hbar^2}{a_0^5}\int_0^\infty dr\, r^2 e^{-2r/a_0}$

$$\underset{r=a_0x/2}{=} \frac{2\hbar^2}{a_0^2}\underbrace{\int_0^\infty dx\, x\, e^{-x}}_{=1} - \frac{\hbar^2}{2a_0^2}\underbrace{\int_0^\infty dx\, x^2 e^{-x}}_{=2} = \left(\frac{\hbar}{a_0}\right)^2.$$

25a) Impuls Erhaltungsgröße $\to \langle \hat{p}\rangle_{\psi(t)} = \langle \hat{p}\rangle_{\psi_0} = p_0$;

nach 13 a) : $\frac{d}{dt}\langle \hat{x}\rangle_{\psi(t)} = \langle \frac{i}{\hbar}[H,\hat{x}]\rangle_{\psi(t)} = \langle \frac{i}{\hbar}[\frac{\hat{p}^2}{2m},\hat{x}]\rangle_{\psi(t)}$

$$= \langle \frac{i}{\hbar}\frac{\hat{p}}{m}\underbrace{[\hat{p},\hat{x}]}_{=-i\hbar}\rangle_{\psi(t)} = \frac{1}{m}\langle \hat{p}\rangle_{\psi(t)} = \frac{p_0}{m}$$

$\to \langle \hat{x}\rangle_{\psi(t)} = x_0 + \frac{p_0}{m}t$

25 b) 3D: gelte $\langle \vec{\hat{X}} \rangle_{\psi_0} = \vec{X}_0$, $\langle \vec{\hat{P}} \rangle_{\psi_0} = \vec{P}_0$.

dann $\langle \vec{\hat{X}} \rangle_{\psi(t)} = \vec{X}_0 + \frac{\vec{P}_0}{m} t$

denn: $\hat{P}_1, \hat{P}_2, \hat{P}_3$ Erhaltungsgrößen $\rightarrow$ $\langle \hat{P}_\ell \rangle_{\psi(t)} = \langle \hat{P}_\ell \rangle_{\psi(0)} = (\vec{P}_0)_\ell$

$$\frac{d}{dt} \langle \hat{X}_\ell \rangle_{\psi(t)} = \langle \frac{i}{\hbar} [H, \hat{X}_\ell] \rangle_{\psi(t)} = \langle \frac{i}{\hbar} \frac{1}{2m} \underbrace{[\hat{P}_1^2 + P_2^2 + P_3^2, \hat{X}_\ell]}_{2P_\ell \cdot [\hat{P}_\ell, \hat{X}_\ell] = -2i\hbar P_\ell} \rangle_{\psi(t)} =$$

$$= \frac{1}{m} \langle \hat{P}_\ell \rangle_{\psi(t)} = \frac{1}{m} (\vec{P}_0)_\ell$$

$$\rightarrow \langle \hat{X}_\ell \rangle_{\psi(t)} = (\vec{X}_0)_\ell + \frac{1}{m} (\vec{P}_0)_\ell \cdot t \quad \text{für } \ell = 1, 2, 3 .$$

26 a)
$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle = \int \frac{dk}{2\pi} \tilde{\psi}(k) e^{-i \frac{\hat{P}^2}{2m} t/\hbar} |\chi_k\rangle$$

$$\underset{\hat{P}|\chi_k\rangle = \hbar k |\chi_k\rangle}{=} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\psi}(k) e^{-i \frac{\hbar k^2}{2m} t} |\chi_k\rangle$$

b)
$$\psi(x,t) = \langle \varphi_x | \psi(t) \rangle \overset{a)}{=} \int \frac{dk}{2\pi} \tilde{\psi}(k) e^{-i \frac{\hbar k^2}{2m} t} \underbrace{\langle \varphi_x | \chi_k \rangle}_{= e^{ikx}} .$$

c)
$$\psi(x) = (4\pi \sigma_0^2)^{1/4} \int \frac{dk}{2\pi} e^{-\frac{\sigma_0^2 k^2}{2}} e^{ikx} = \frac{(4\pi \sigma_0^2)^{1/4}}{2\pi} \sqrt{\frac{2\pi}{\sigma_0^2}} e^{-\frac{x^2}{2\sigma_0^2}}$$

$$= \frac{1}{(\pi \sigma_0)^{1/4}} e^{-x^2/2\sigma_0^2} ;$$

$$\psi(x,t) \overset{b)}{=} (4\pi \sigma_0^2)^{1/4} \int \frac{dk}{2\pi} e^{-\underbrace{\left(\frac{\sigma_0^2}{2} + i \frac{\hbar}{2m} t\right)}_{\frac{\sigma_0^2}{2}\left(1 + i \underbrace{\frac{\hbar}{m\sigma_0^2}}_{1/\tau} t\right) = \frac{\sigma_0^2}{2}(1 + it/\tau)} k^2 + ikx}$$

$$= \frac{(4\pi \sigma_0^2)^{1/4} \sqrt{2\pi}}{2\pi \sqrt{\sigma_0^2 |1 + it/\tau|}} e^{-\frac{x^2}{2\sigma_0^2 (1 + it/\tau)}} =$$

$$= \frac{1}{(\pi \sigma_0^2 |1 + it/\tau|^2)^{1/4}} e^{+i \frac{x^2 t}{2\sigma_t^2}} e^{-\frac{x^2}{2\sigma_t^2}}$$

$$\frac{1}{1 + it/\tau} = \frac{1 - it/\tau}{1 + t^2/\tau^2}$$

$$\rightarrow |\psi(x,t)|^2 = \frac{1}{\sqrt{\pi \sigma_t^2}} e^{-x^2/\sigma_t^2} .$$