Lösungshinweise Blatt 6

27a) $[H, \hat{x}] = [\hat{p}^2/2m + U(\hat{x}), \hat{x}] = \frac{1}{2m}[\hat{p}^2, \hat{x}] = \frac{\hat{p}}{m}[\hat{p}, \hat{x}] = -i\hbar \hat{p}/m$

$\rightarrow \quad \hat{p} = \frac{im}{\hbar}[H, \hat{x}]$

$\rightarrow \quad \langle \hat{p} \rangle_\phi = \frac{im}{\hbar} \langle \phi | H\hat{x} - \hat{x}H | \phi \rangle = \frac{im}{\hbar} \langle \phi | E\hat{x} - \hat{x}E | \phi \rangle = 0$

(existiert, da ϕ normierbar) $\quad$ $|\phi\rangle$ Energieeigenzustand zur Energie E

b) $\phi_{a/b}$ genügen stat. S.Gl. zur gemeinsamen Energie E:

$$\phi_{a/b}'' = \frac{2m}{\hbar^2}(U - E)\phi_{a/b} \qquad (*)$$

$$\rightarrow \quad \frac{d}{dx}(\phi_a' \phi_b - \phi_a \phi_b') = \phi_a'' \phi_b - \phi_a \phi_b''$$

$$\overset{(*)}{=} \frac{2m}{\hbar^2}(U-E)(\phi_a \phi_b - \phi_a \phi_b) = 0$$

d.h. $\phi_a' \phi_b - \phi_a \phi_b' = \text{konst} = \kappa$; da $\phi_{a/b}(x)$ und $\phi_{a/b}'(x)$ für $x \to \infty$ verschwinden ($\phi_{a/b}$ normierbar!) ist $\kappa = 0$ und somit $\phi_a'/\phi_a - \phi_b'/\phi_b = 0$, d.h. $(\ln \phi_a - \ln \phi_b)' = 0$

$\rightarrow \quad \ln \phi_a - \ln \phi_b = \text{konst.} = \gamma$

$\rightarrow \quad \phi_a(x) = e^\gamma \phi_b(x) \quad \blacksquare$

28 a) $x_{11} = x_{22} = L/2$ (symmetry)

$$x_{12} = \frac{2}{L}\int_0^L dx \, \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) \underset{x \to L\tilde{x}/\pi}{=} \frac{2}{L}\frac{L^2}{\pi^2} \underbrace{\int_0^\pi d\tilde{x} \, \sin(\tilde{x}) \sin(2\tilde{x})}_{= -8/9}$$

$$= -\left(\frac{4}{3\pi}\right)^2 L$$

$$x_{21} = \langle \phi_2 | \hat{x} | \phi_1 \rangle = \langle \phi_1 | \hat{x} | \phi_2 \rangle^* = x_{12}^* = x_{12}$$

$$\rightarrow \quad \underline{\underline{X}} = \frac{L}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{4}{3\pi}\right)^2 L \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$p_{11} = p_{22} = 0$ nach 27 a)

$$p_{21} = -i\hbar \frac{2\pi}{L^2} \int_0^L dx \, \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) \underset{x = L\tilde{x}/\pi}{=} -i\frac{\hbar 2}{L} \underbrace{\int_0^\pi d\tilde{x} \, \sin(2\tilde{x}) \cos(\tilde{x})}_{= 4/3} = -i\frac{8\hbar}{3L}$$

$$p_{12} = p_{21}^* = +i\frac{8\hbar}{3L}$$

$$\rightarrow \quad \underline{\underline{P}} = \frac{8\hbar}{3L}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$$

$$H_{ij} = E_i \delta_{ij} \quad : \quad \underline{\underline{H}} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix} = \frac{\hbar^2\pi^2}{2mL^2}\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$$

18 b) in Komponenten bzgl. $|\phi_1\rangle, \; |\phi_2\rangle$:

$$\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \rightarrow \psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-iE_1t/\hbar}\\ e^{-E_2t/\hbar}\end{pmatrix} = \frac{e^{-iE_2t/\hbar}}{\sqrt{2}}\begin{pmatrix}e^{i\Omega t}\\ 1\end{pmatrix}$$

mit $\Omega = (E_2 - E_1)/\hbar$

$$\rightarrow \quad \langle x\rangle_{\psi(t)} = \psi(t)^{+} \, \underline{\underline{X}} \, \psi(t)$$

$$\overset{(a)}{=} L\,\psi^{+}(t)\left(\frac{1}{2}\mathbb{1} - \left(\frac{4}{3\pi}\right)^2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\psi(t)$$

$$= \frac{L}{2} - \left(\frac{4}{3\pi}\right)^2 L\left(\frac{e^{+i\Omega t} + e^{-i\Omega t}}{2}\right)$$

$$= \frac{L}{2} - \left(\frac{4}{3\pi}\right)^2 L\cos(\Omega t)$$

$$\langle \hat{p}\rangle_{\psi(t)} = \psi^{+}(t)\,\underline{\underline{P}}\,\psi(t) = \frac{8\hbar}{3L}\psi^{+}(t)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\psi(t)$$

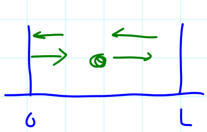
$$= \frac{8\hbar}{3a}\left(\frac{i e^{-i\Omega t} - i e^{i\Omega t}}{2}\right) = \frac{8\hbar}{3a}\sin(\Omega t)$$

$$\langle H\rangle_{\psi(t)} = \langle H\rangle_{\psi(0)} = \psi^{+}(0)\,\underline{\underline{H}}\,\psi(0) = \frac{1}{2}(E_1 + E_2) \,.$$

29) QM: $F = -\frac{d}{dL}E_n(L) = -\frac{d}{dL}\left(\frac{\hbar^2\pi^2}{2m}\frac{n^2}{L^2}\right) = \frac{2}{L}\underbrace{\left(\frac{\hbar^2\pi^2 n^2}{2mL^2}\right)}_{E_n(L)}$

also $F = \frac{2E_n(L)}{L}$.

klassisch : $E = p_0^2/2m$



Periodendauer $T = 2\frac{L}{v} = \frac{2mL}{p_0}$

$\rightarrow$ mittlere Kraft $F = \frac{\Delta P}{\Delta T} = \frac{2p_0}{T} = \frac{1}{L}\frac{p_0^2}{m} = \underline{\underline{\frac{2E}{L}}}$!

30) $\phi(x) = \alpha\, e^{-k_\varepsilon |x|}$ mit $k = \sqrt{2m\varepsilon}/\hbar$, $\varepsilon = -E$

Lsg der st. S.Gl. für $x \neq 0$, stetig in $x = 0$;

$$\delta\phi' = \phi'(a) - \phi'(-a) = \int_{-a}^{a} \phi''(x)\, dx \underset{\phi'' = \frac{2m}{\hbar^2}(u(x) - E)\phi}{=} \frac{2m}{\hbar^2} \int_{-a}^{a} (-u\,\delta(x) + \varepsilon)\,\phi(x)\, dx$$

$$= -\frac{2mu}{\hbar^2}\,\phi(0) \qquad (\text{im Limes } a \to 0)$$

für $\phi(x) = \alpha\, e^{-k_\varepsilon |x|}$ folgt:

$$-\frac{2mu}{\hbar^2}\,\alpha = -2\alpha k_\varepsilon$$

$$\rightsquigarrow \quad \underset{\sqrt{2m\varepsilon}/\hbar}{\overset{\parallel}{k_\varepsilon}} = \frac{mu}{\hbar^2} \qquad \rightsquigarrow \quad -E = \varepsilon = \frac{mu^2}{2\hbar^2}$$