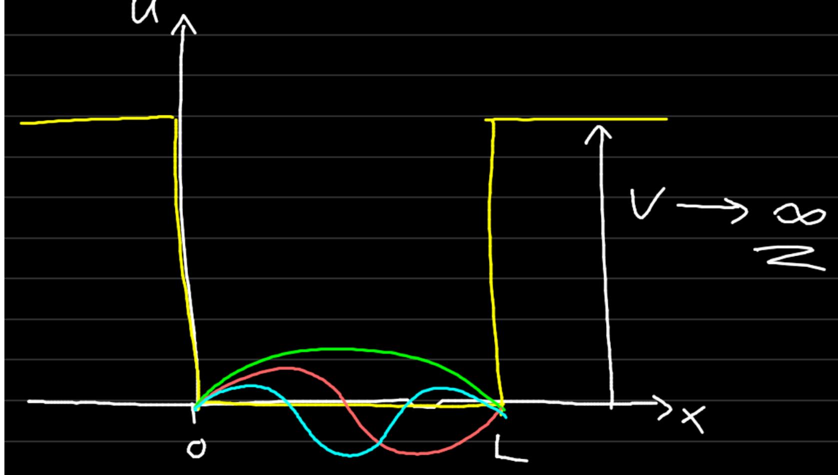
Wdhlg.: "Teilchen im Kasten"



→ $\phi(x)$ für $x \in [0, L]$ Lsg. der freien stat. S.Gl mit R.B $\phi(0) = \phi(L) = 0$

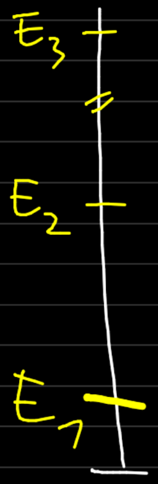
→ $$\phi'' = -\overbrace{\frac{2m}{\hbar^2}E}^{k_E^2}\,\phi$$

→ $$\phi_n(x) = \sqrt{\frac{2}{L}}\,\sin(k_n x)$$

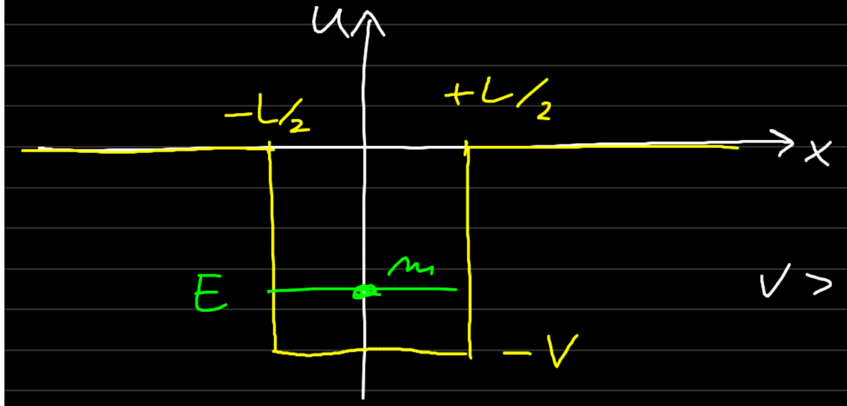
$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

$$k_n = \frac{\pi}{L}\cdot n\,, \quad n \in \mathbb{N}$$

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2}\cdot n^2$$



Teilchen im Potenzialtopf



$V > 0$

ges: gebundene Teilchenzust. (= Energieeigenzust.)

$\rightarrow$ $\underline{-V < E < 0}$

st. S.G.:

für $|x| > L/2$:

$$\phi''(x) = -\frac{2m}{\hbar^2} E \,\phi(x)$$

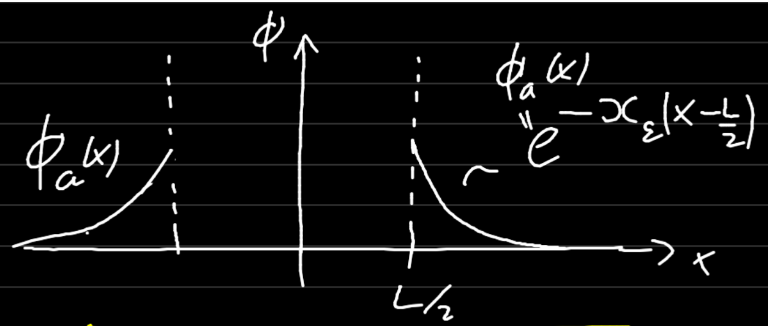
$$0 < \varepsilon := -E : \quad \phi'' = \underbrace{\frac{2m\varepsilon}{\hbar^2}}_{\varkappa_\varepsilon^2} \phi(x)$$

$$\varkappa_\varepsilon = \sqrt{2m\varepsilon}/\hbar$$

$$\rightarrow \boxed{\phi''(x) = \varkappa_\varepsilon^2 \,\phi(x)}$$

→ normierbare spezielle Lsgen:

$|x| > L/2$: $\boxed{\phi_a(x) = e^{-\varkappa_\varepsilon(|x|-L/2)}}$



<u>stat. S. Gl. für $|x| \le L/2$</u>

$$\phi'' = \frac{2m}{\hbar^2}(U(x)-E)\,\phi(x)$$

$$= \frac{2m}{\hbar^2}(-V+\varepsilon)\,\phi(x)$$

$$\phi''(x) = -\underbrace{\frac{2m}{\hbar^2}(V-\varepsilon)}_{k_\varepsilon^2}\,\phi(x)$$

$$k_\varepsilon = \sqrt{2m(V-\varepsilon)}/\hbar$$

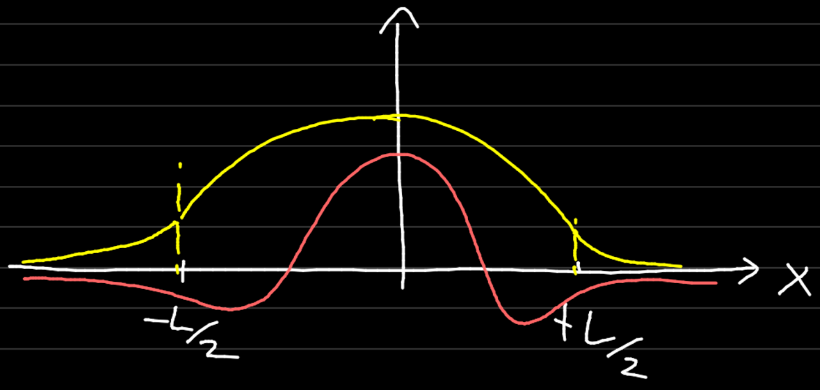
→ $\boxed{\phi''(x) = -k_\varepsilon^2\,\phi(x)}$

spezielle Lsgen:

$$\boxed{\begin{aligned}\phi_b(x) &= \sin(k_\varepsilon x)\\ \phi_\ell(x) &= \cos(k_\varepsilon x)\end{aligned}}$$

globale symmetrische Lösungen:

Ansatz: $$\phi(x) = \begin{cases} \alpha\, e^{-\varkappa_\varepsilon(|x|-L/2)} & : |x| > \frac{L}{2} \\ \beta \cos(k_\varepsilon x) & : |x| \le \frac{L}{2} \end{cases}$$



ϕ, ϕ' stetig, insb. $x = \pm L/2$

$$\boxed{\begin{aligned} \alpha &= \beta \cos(k_\varepsilon L/2) \quad \mathrm{I} \\ \varkappa_\varepsilon \alpha &= \beta k_\varepsilon \sin(k_\varepsilon L/2) \quad \mathrm{II} \end{aligned}}$$

II/I:

$$\boxed{\frac{\varkappa_\varepsilon}{k_\varepsilon}} = \frac{\sin(k_\varepsilon L/2)}{\cos(k_\varepsilon L/2)} = \boxed{\tan(k_\varepsilon L/2)}$$

mit $\quad x_\varepsilon = \sqrt{2m\varepsilon}/\hbar$

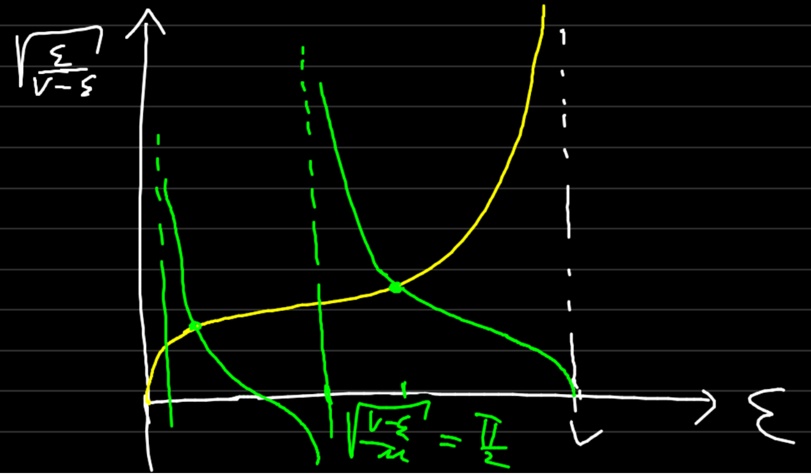
$\quad\quad h_\varepsilon = \sqrt{2m(V-\varepsilon)}/\hbar$

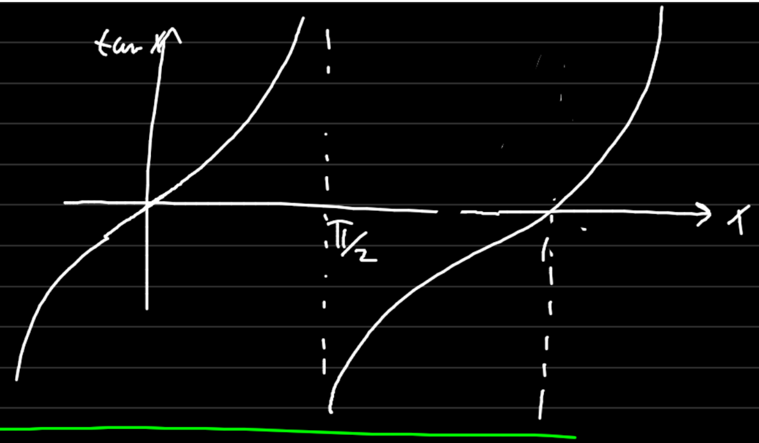
also: $\quad \sqrt{\frac{\varepsilon}{V-\varepsilon}} = \tan\left(\underbrace{\frac{\sqrt{2m(V-\varepsilon)}}{\hbar}\frac{L}{2}}_{\sqrt{\frac{V-\varepsilon}{\underbrace{2\hbar^2/mL^2}_{u}}}}\right)$

$u := \frac{2\hbar^2}{m}\cdot\frac{1}{L^2}$

$\to \boxed{\sqrt{\frac{\varepsilon}{V-\varepsilon}} \overset{!}{=} \tan\left(\sqrt{\frac{V-\varepsilon}{u}}\right)}$

Lösungen $\to$ Eigenenergien!





1) gibt es immer (sym.) Lsg?

Ja!

2) Wieviele Lösungen?

Anzahl Lösungen =

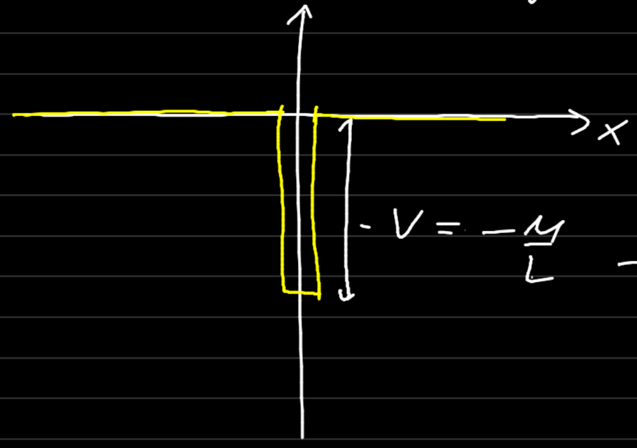
Anzahl Nullstellen von $\tan\left(\sqrt{\frac{V-\varepsilon}{m}}\right)$ für $\varepsilon \in [0, V] \Rightarrow$

$$= \left\lceil \frac{\sqrt{V/m}}{\pi} \right\rceil \quad \checkmark$$

$\lceil x \rceil :=$ kleinste ganze Zahl $\geq x$

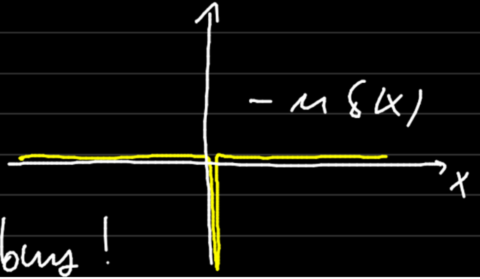
für $L \to 0$: $\rightsquigarrow$ $u \to \infty$

$\rightsquigarrow$ genau eine Lösung!



$-V = -\frac{u}{L}$ $\rightsquigarrow$ $V = L u$

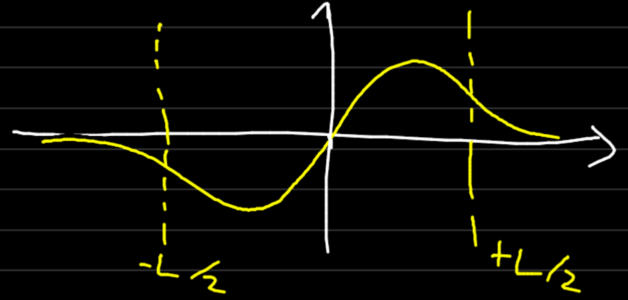
$\rightsquigarrow$ $\boxed{u(x) = -u\,\delta(x)}$



$\hookrightarrow \exists$ geb. Zust $\rightsquigarrow$ vgl. Übung!

globale antisymm. Lsgen:

$$\phi(x) = \begin{cases} \alpha \operatorname{sgn}(x)\, e^{-\varkappa_\varepsilon(|x| - L/2)} & : |x| > \frac{L}{2} \\ \beta \sin(k_\varepsilon x) & : |x| < \frac{L}{2} \end{cases}$$

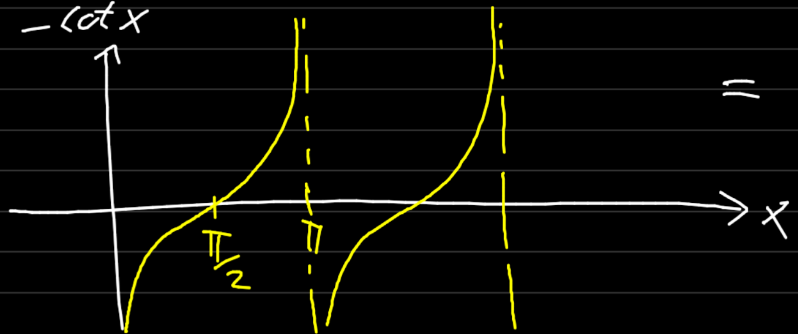


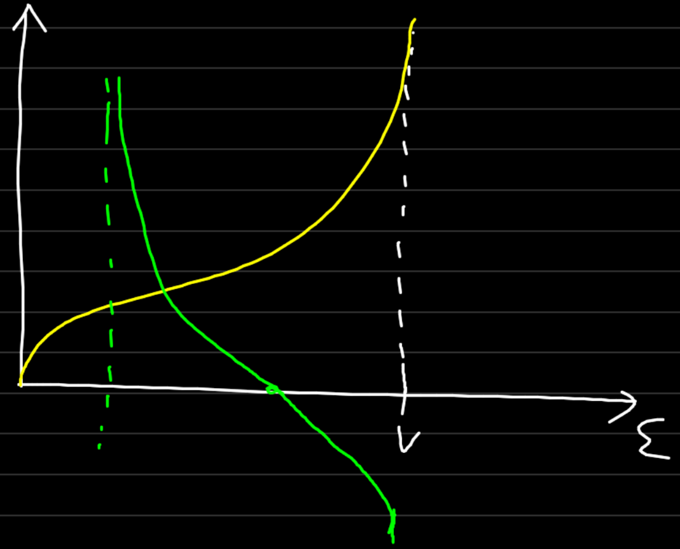
ϕ, ϕ' stetig in $x = +L/2$:

$$\begin{cases} \mathrm{I} & \alpha = \beta \sin(k_\varepsilon L/2) \\ \mathrm{II} & -\alpha \varkappa_\varepsilon = \beta k_\varepsilon \cos(k_\varepsilon L/2) \end{cases}$$

II/I: $$\boxed{\sqrt{\frac{\varepsilon}{V-\varepsilon}} \stackrel{!}{=} -\frac{\cos(k_\varepsilon L/2)}{\sin(k_\varepsilon \frac{L}{2})} = -\cot\left(k_\varepsilon \frac{L}{2}\right)}$$

$$= -\cot\left(\sqrt{\frac{V-\varepsilon}{\mu}}\right)$$



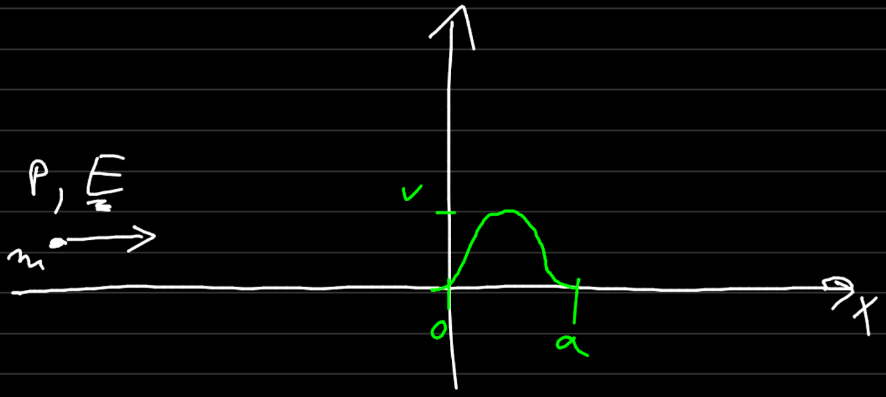


- Lösg für $\sqrt{V/\mu} > \pi/2$!
- Keine Lsg für $\sqrt{V/\mu} < \frac{\pi}{2}$!

Reflexion und Transmission (1D) eines Teilchens an Potentialschwelle

↳ a) Tunneleffekt

b) Streuexperimente :



klassisch: $E > V$: Transmission

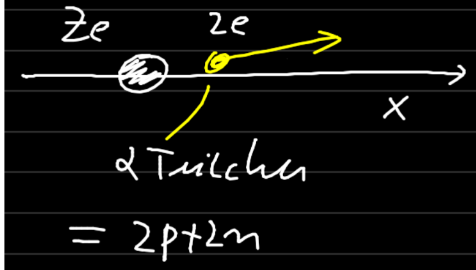
$E < V$: Reflexion

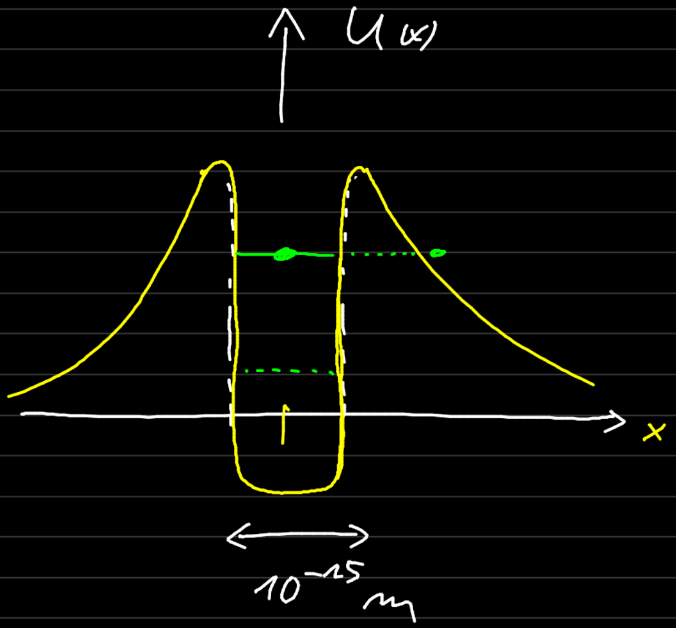
q.m.: $E > V$: Transmission mit Wkt. $\underline{< 1}$

$E < V$: Transmission mit Wkt $\underline{> 0}$.

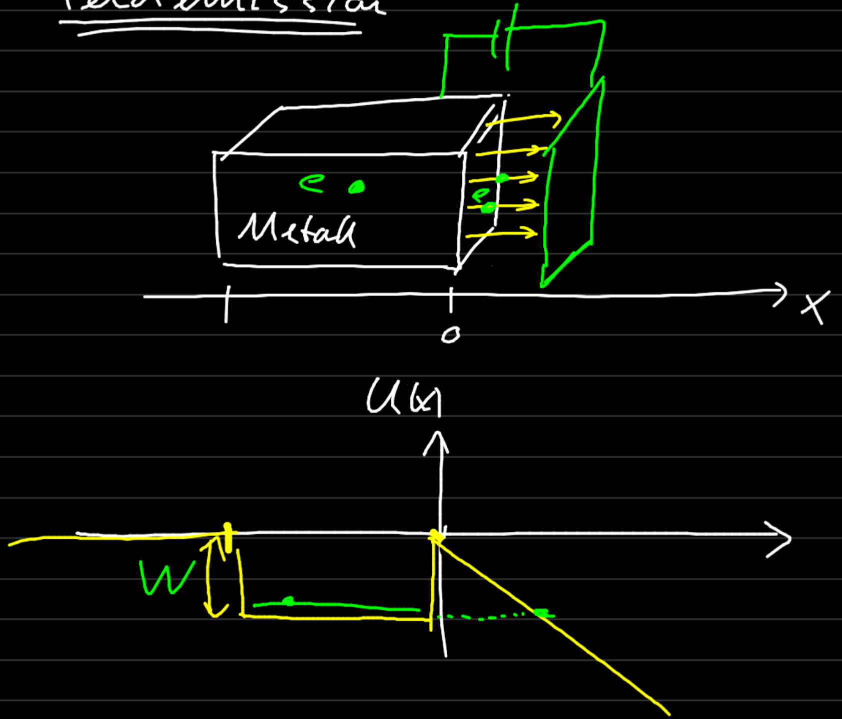
Tunneleff relevant z.B für.

α-Zerfall

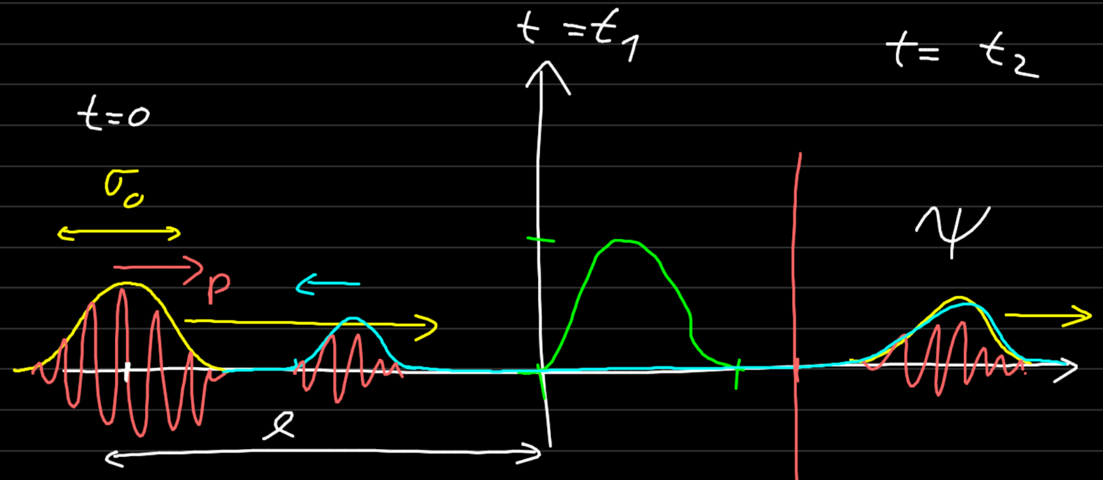




Feldemission



q. m. Beschreibung : naiv :



$$\psi_0(x) = \dots\, e^{-(x+l)^2/2\sigma_0^2}\; e^{i\frac{p}{\hbar}x}$$

$$\xrightarrow{S.G.} \psi(x,t)$$