Streuansatz: Teilchen: Masse m

Impuls $p = \hbar k > 0$

$\hookrightarrow$ Energie $E = \frac{\hbar^2 k^2}{2m}$

$$\phi(x) = \begin{cases} 1\, e^{ikx} + \underline{\underline{r}}\, e^{-ikx} & : x < 0 \\ \phi_{\underline{\underline{s,u}}}(x) & : 0 \leq x \leq d \\ \underline{\underline{t}}\, e^{ikx} & : x > d \end{cases}$$

mit $\phi_{s,u}(x)$ allg. Lsg. der st. S.G. für $0 \leq x \leq d$!

$$k = \sqrt{2mE}/\hbar$$

Parameter r, t, s, u bestimmt durch Stetigkeit von ϕ, ϕ' insbes. an $x = 0$, $x = d$!

$\Rightarrow$ Transmissionswkt $T = |t|^2$

Reflexionswkt $R = |r|^2 = (1 - T)$

Exkurs: Whts.-Stromdichte

Aufenthaltswkt.dichte:

$$\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$$

$\leadsto$ genügt: $i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$

Whts-erhaltung:

$$\frac{d}{dt}\int_{\mathbb{R}^3} d^3\vec{x}\;\rho(\vec{x},t) = 0$$

$G \subset \mathbb{R}^3$

$$\frac{d}{dt}\int_G d^3x\;\rho = -\int_{\partial G}\vec{j}\,d\vec{f}$$

für alle G!

$\leadsto$ Kontinuitätsgl.: $\boxed{\dot{\rho} + \operatorname{div}\vec{j} = 0}$

$\boxed{\vec{j} = ?}$

$$\dot{\rho} = \frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}(\psi^* \psi)$$

$$= \dot{\psi}^* \psi + \psi^* \dot{\psi}$$

$$= -\frac{i\hbar}{2m}(\Delta\psi^*)\psi + \cancel{\frac{i}{\hbar}U\psi^*\psi} + \frac{i\hbar}{2m}\psi^*\Delta\psi - \cancel{\frac{i}{\hbar}U\psi^*\psi}$$

$$\dot{\psi} = \frac{i\hbar}{2m}\Delta\psi - \frac{i}{\hbar}U\psi$$

$$= \frac{i\hbar}{2m}\left(-(\Delta\psi^*)\psi + \psi^*\Delta\psi\right)$$

$$= \frac{i\hbar}{2m}\Big(\underbrace{-\mathrm{div}(\psi\,\mathrm{grad}\,\psi^*) + \mathrm{div}(\psi^*\,\mathrm{grad}\,\psi)}_{2i\,\mathrm{Im}\,\mathrm{div}(\psi^*\,\mathrm{grad}\,\psi)}\Big)$$

$$= -\frac{\hbar}{m}\,\mathrm{Im}\,\mathrm{div}(\psi^*\,\mathrm{grad}\,\psi)$$

d.h. $\dot{\rho} = -\mathrm{div}\,\vec{j}$ mit $\boxed{\vec{j} = \frac{\hbar}{m}\,\mathrm{Im}\,\psi^*\,\mathrm{grad}\,\psi}$

Wkts-Stromdichte

in 1D: Konti-gl.: $\dot{\varrho} + \frac{\partial}{\partial x} j = 0$

falls $\psi(x,t)$ stationär $\dot{\varrho} = 0$

$\to 0 = \frac{\partial}{\partial x} j \to j$ konst.!

Bsp: $\chi_k(x) = e^{ikx}$

$$\Rightarrow j = \frac{\hbar}{m} \mathrm{Im}\left(e^{-ikx}\, ik\, e^{ikx}\right) = \frac{\hbar k}{m}$$

$t e^{ikx}$: $j = |t|^2 \frac{\hbar k}{m}$

$$j = \frac{\hbar}{m} \mathrm{Im}\left(\psi^* \frac{\partial}{\partial x} \psi\right)$$

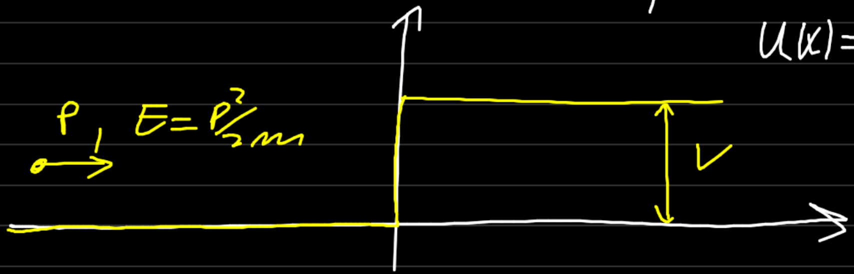
$1 e^{ikx} + r e^{-ikx}$:

$$\Rightarrow j = \frac{\hbar k}{2m}\left(1 - |r|^2\right)$$

$$\Rightarrow 1 - |r|^2 = |t|^2$$

$\to$ Transm. wkt $T = |t|^2 = 1 - |r|^2 = 1 - R$

Beispiel: Reflexion/Transm. an der Potentialstufe



$$U(x) = \begin{cases} 0 & : x < 0 \\ V & : x \geq 0 \end{cases}$$

Streuansatz

$$\phi(x) = \begin{cases} 1 e^{ikx} + r e^{-ikx} & : x < 0 \\ t e^{i\tilde{k}x} & : x \geq 0 \end{cases}$$

$$k = \sqrt{2mE}/\hbar$$

$$\tilde{k} = \sqrt{2m(E-V)}/\hbar$$

Stetigkeit von ϕ, ϕ' in $x = 0$:

I: $1 + r = t$

II: $ik - ikr = i\tilde{k}t$

Iik: $k - kr = \hat{k} + \tilde{k}r$

$$\Leftrightarrow \boxed{r = \frac{k - \tilde{k}}{k + \tilde{k}}}$$

1. Fall : $E < V$:

$$\tilde{k} = i\underbrace{\sqrt{2m(V-E)}}_{q} = iq$$

$$\underline{\underline{R}} = |r|^2 = \left|\frac{k-iq}{k+iq}\right|^2 = \underline{\underline{1}}$$

$$r \overset{!}{=} e^{i2\vartheta}$$

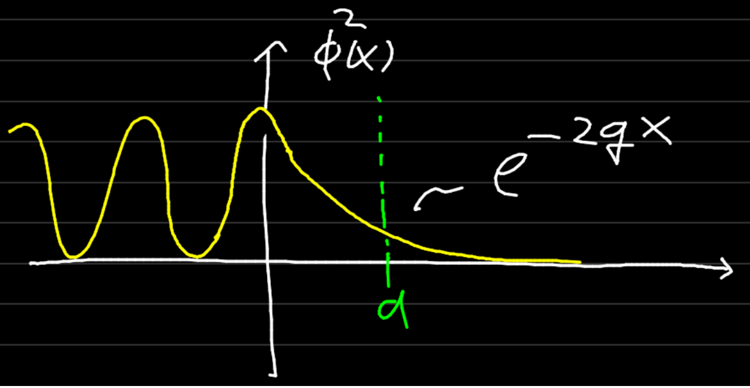
$\underline{\underline{x<0}}$:

$$\phi(x) = e^{ikx} - e^{i2\vartheta}e^{-ikx}$$

$$= e^{i\vartheta}\left(e^{i(kx-\vartheta)} - e^{-i(kx-\vartheta)}\right)$$

$\underline{\underline{x<0}}$: $\phi(x) = 2i\cancel{e^{i\vartheta}}\sin(kx-\vartheta)$

$$\leadsto \phi(x) = \begin{cases} 2\sin(kx-\vartheta) & : x<0 \\ -2\sin(\vartheta)\,e^{-qx} & : x \geq 0 \end{cases}$$



2. Fall: $E > V$:

$$k = \sqrt{2mE}/\hbar$$

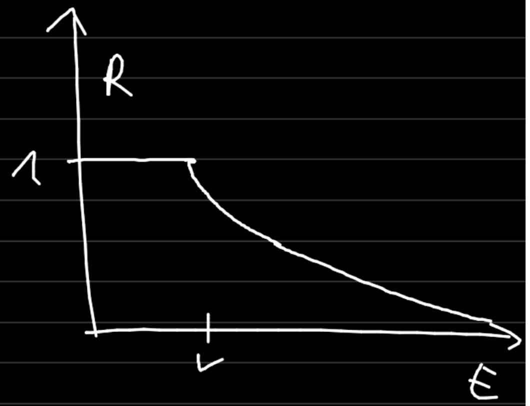
$$\tilde{k} = \sqrt{2m(E-V)}/\hbar$$

$$\Rightarrow r = \frac{k - \tilde{k}}{k + \tilde{k}} = \frac{\sqrt{E} - \sqrt{E-V}}{\sqrt{E} + \sqrt{E-V}}$$

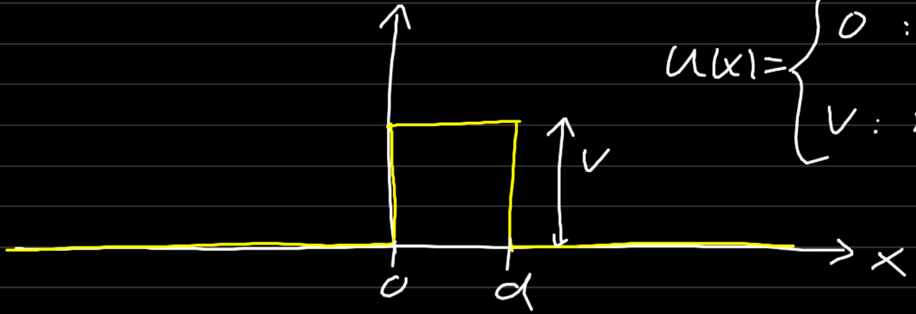
$$R = |r|^2 = \left|\frac{1 - \sqrt{1 - V/E}}{1 + \sqrt{1 - V/E}}\right|^2 < 1\ !$$

$$R = \begin{cases} 1 - 2V/E & : E \gg V \\ 1 - 4\sqrt{\frac{E-V}{V}} & : \underbrace{(E-V)}_{>0} \ll V \end{cases}$$

$$R = \left|\frac{1 - \sqrt{\frac{E-V}{V}}}{1 + \sqrt{\frac{E-V}{V}}}\right|^2$$



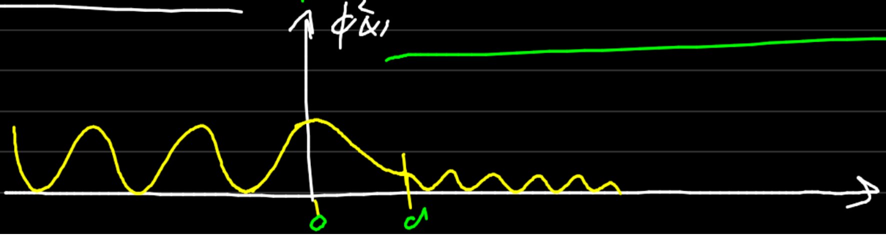
Refl. / Trans. an Potenzialschwelle:



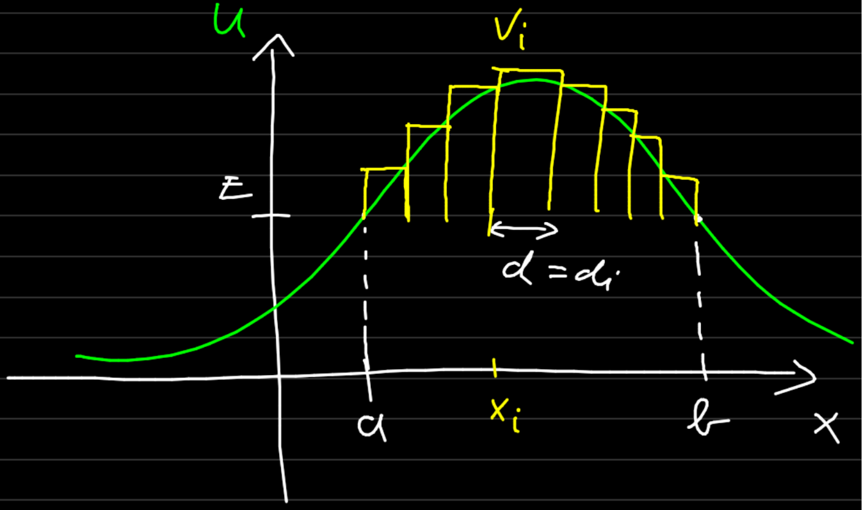
$$U(x) = \begin{cases} 0 & : x<0,\ x>d \\ V & : x \in [0,d] \end{cases}$$

Transmission

$$\boxed{T = |t|^2 \simeq e^{-2q d} \quad !}$$



Tunneleffekt für allg. Potenzialsch. (in Gamov-Näherung):



$$V_i = U(x_i)$$

$$g_i = \sqrt{2m(V_i - E)}/\hbar = \sqrt{2m(U(x_i)-E)}/\hbar$$

$$\to T_i \simeq e^{-2g_i d}$$

Tunnelw. für i-te Barriere

$\to$ Tunnelwkt für gesamte Barr.:

$$T = \prod_{i=1}^{N} e^{-2g_i d} = \exp\left(-\sum_{i=1}^{N} 2 g_i d\right)$$

$$T = \exp\left(-\underbrace{\sum_{i=1}^{N} \sqrt{8m(U(x_i)-E)}\,\frac{d}{\hbar}}_{d\to 0}\right)$$

$$\boxed{T = \exp\left(-\frac{1}{\hbar}\int_a^a \sqrt{8m(U(x)-E)}\,dx\right)}$$