Wdhlg.: Aufenthaltswkt. $\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$

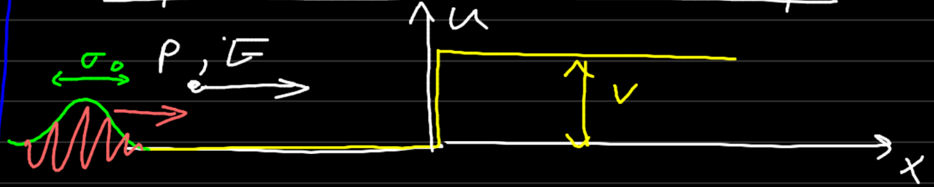
S.Gl., Wktserhaltung

→ **Wktsstromdichte**

$$\vec{j}(\vec{x},t) = \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^* \,\mathrm{grad}\,\psi\right)$$

→ **Kont.-gl.:** $$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$$

Refl. / Trans. an Pot.-stufe



Streuansatz

einfall. Teilchen

refl. Teilchen

$$\phi(x) = \begin{cases} e^{ikx} + r\,e^{-ikx} & : x<0 \\ t\,e^{ikx} & : x \geq 0 \end{cases}$$

trans. Teilchen

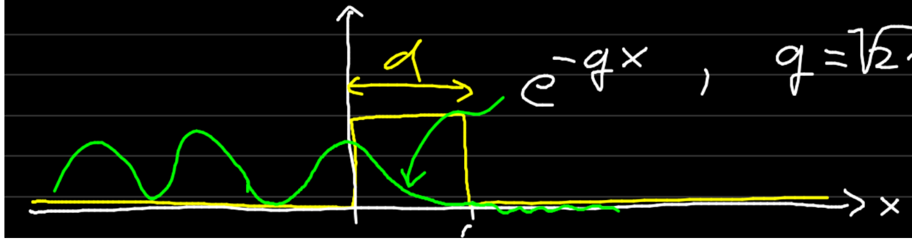
+ St. S.Gl. + Stetigkeit

→ r, t

→ Reflexionswkt $R = |r|^2$:

- $E \le V$: $R = 1$
- $E > V$: $R = \left|\dfrac{1-\sqrt{1-V/E}}{1+\sqrt{1-V/E}}\right|^2 = \begin{cases} 1-4\sqrt{1-V/E} & : \boxed{E-V \ll V} \\ \left(\dfrac{V}{2E}\right)^2 & : \boxed{E \gg V} \end{cases}$

Potenzialschwelle:



e^{-qx}, $q = \sqrt{2m(V-E)}/\hbar$

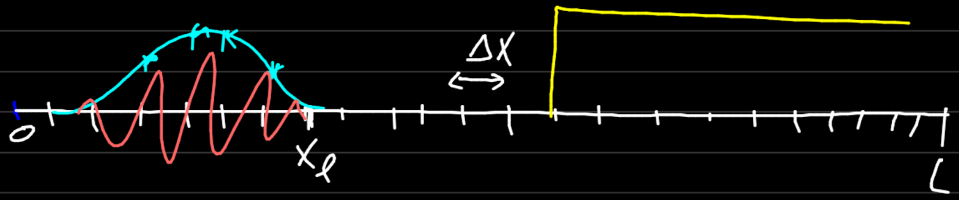
$t \simeq e^{-qd}$

auch für $\underline{\underline{E<V}}$ Transmission mit Wkt. $T \simeq e^{-2qd} > 0$!

Tunneleffekt

(F. Hund 1927, Gamov 1928, ...)

Veranschaulichung des Streuansatzes: Numerik!



$$\psi_\ell = \psi(x_\ell)$$

$$\hookrightarrow \psi_\ell' = \psi'(x_\ell) = (\psi_{\ell+1} - \psi_\ell)\, \frac{1}{\Delta x}$$

$$\psi_\ell'' = \psi''(x_\ell) = (\psi_{\ell+1} - 2\psi_\ell + \psi_{\ell-1})\, \frac{1}{\Delta x^2}$$

$$\psi(x,t) \xrightarrow{\Delta t} \psi(x, t+\Delta t)$$

$$\{\psi_\ell(t)\} \xrightarrow{\Delta t} \{\psi_\ell(t+\Delta t)\}_\ell$$

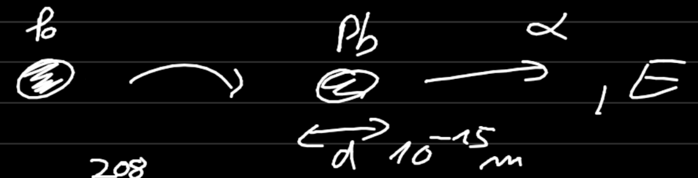
$$\psi(x_\ell, t+\Delta t) \cong \psi(x_\ell, t) + \dot{\psi}(x_\ell, t)\cdot \Delta t$$

$$\downarrow$$

$$\psi_\ell(t+\Delta t) = \psi_\ell(t) + \dot{\psi}_\ell(t)\,\Delta t$$

$$\dot{\psi}_\ell = \frac{i}{\hbar}\left(\frac{\hbar^2}{2m}\psi_\ell'' - U_\ell\, \psi_\ell\right)$$

α-Zerfall vom Polonium zu Blei



1) $^{212}_{84}Po \longrightarrow {}^{208}_{82}Pb + \alpha$, $E_1 = 8{,}95\,MeV$

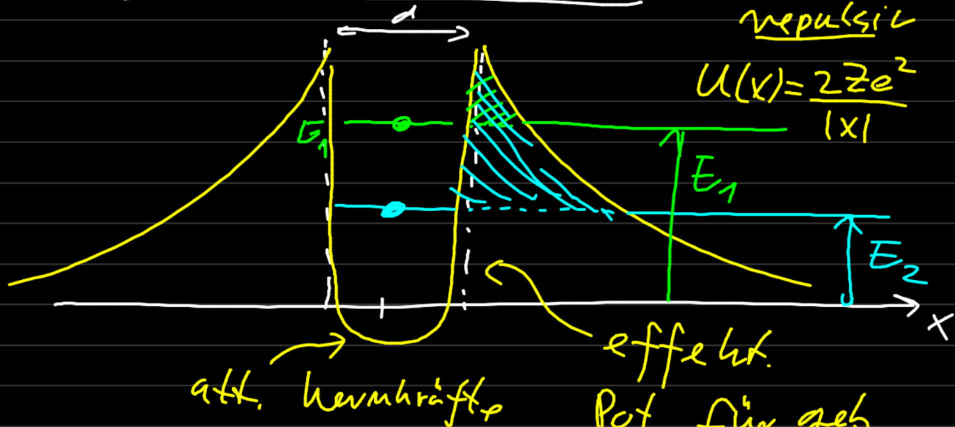
$\tau_1 = 3 \cdot 10^{-7}\,s$

2) $^{210}_{84}Po \longrightarrow {}^{206}_{82}Pb + \alpha$, $E_2 = 5{,}41\,MeV$

$\tau_2 = 1{,}3 \cdot 10^{+7}\,s$

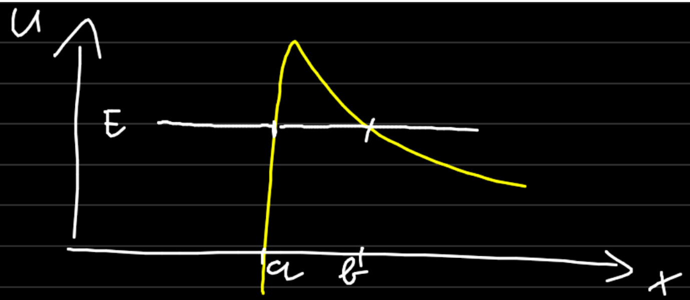
$$\frac{\tau_2}{\tau_1} = \underline{\underline{10^{14}}} \quad ?!$$

Gamovs Modell: (10)



Tunnel-WK für allg. Pot.schwelle:

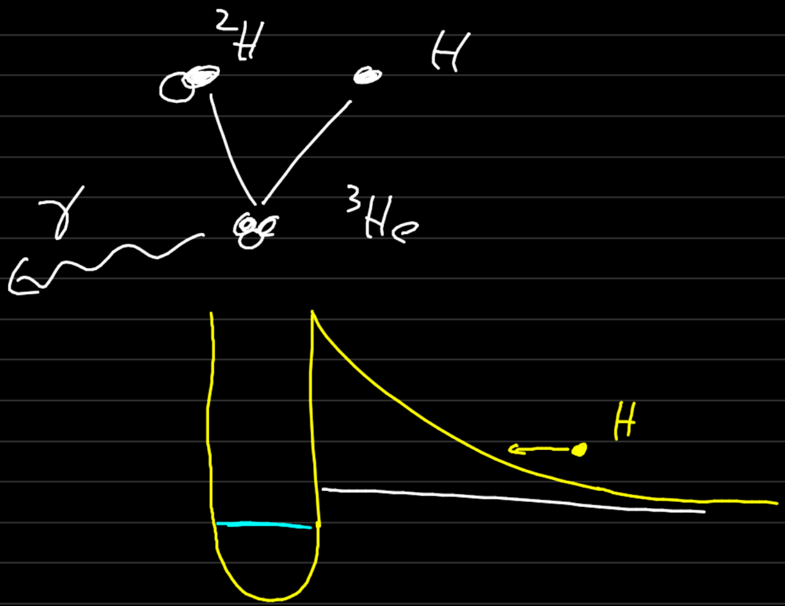
(T. Fließbach, Q11)



$$\to T \simeq \exp\left(-\frac{1}{\hbar}\int_a^b \sqrt{8m(U(x)-E)}\,dx\right)$$

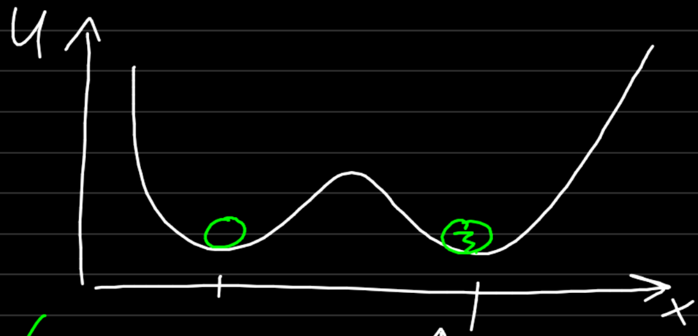
$$\to \frac{\tau_1}{\tau_2} = \frac{T_2}{T_1} = \begin{cases} 4\cdot 10^{14} & ! \text{ (Theorie)} \\ 2\cdot 10^{14} & \text{Exp.} \end{cases}$$

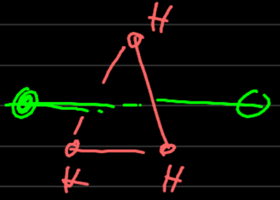
ähnlich: Kernfusion

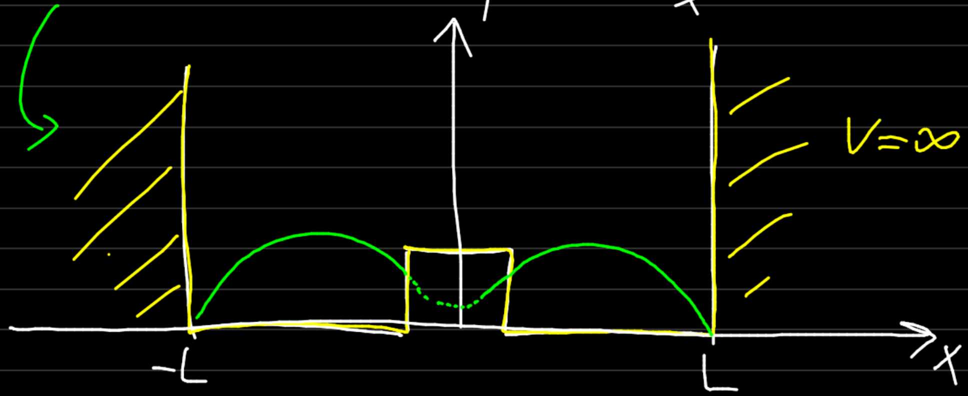


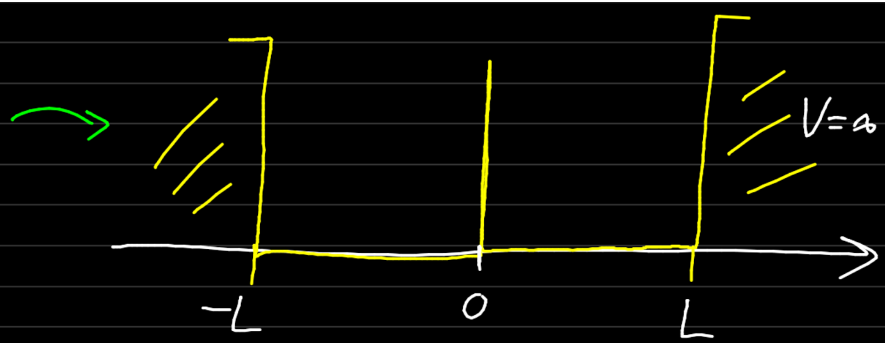
Teilchen im Doppelmuldenpot.

$(\to NH_3)$









$$u(x)=\begin{cases}\infty & : |x|>L\\ u\,\delta(x) & : |x|\le L\end{cases}$$

ges.: Lsg. $\phi(x)$ der stat. S.Gl.
$x\in[-L,L]$

mit Randbed. $\phi(\pm L)=0$

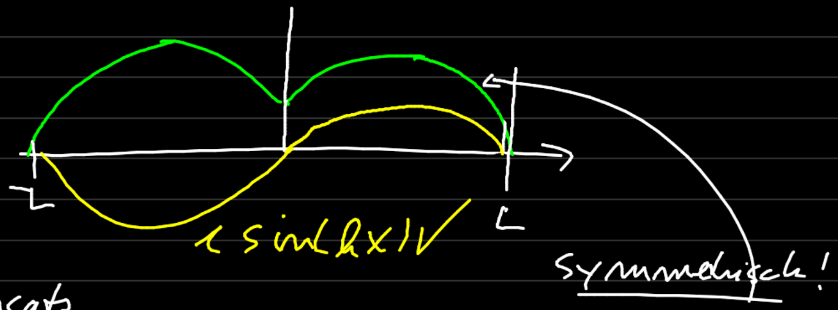
$$\phi''(x) = \frac{2m}{\hbar^2}\left(u\,\delta(x) - E\right)\phi$$

$x \neq 0$: $\phi''(x) = -k^2\phi(x)$, $\quad k = \sqrt{2mE}/\hbar$

$\hookrightarrow$ $\phi(x)$: $e^{\pm ikx}$, $\cos(kx)$, $\sin(kx)$, $\ldots$, $\sin(kx+b)$

$x = 0$: Unstetigkeit in ϕ' bei $x=0$!

$$\boxed{\delta\phi'_{\varepsilon\to 0^+} = \phi'(\varepsilon) - \phi'(-\varepsilon) = \int_{-\varepsilon}^{\varepsilon}\phi''(x)\,dx = \int_{-\varepsilon}^{\varepsilon}\left(\frac{2m}{\hbar^2}u\,\delta(x) - \frac{2mE}{\hbar^2}\right)\phi(x)\,dx = \frac{2m}{\hbar^2}u\,\phi(0)} \quad (*)$$



Ansatz: $\phi(x) = A\sin(k(|x|+b))$

$\hookrightarrow$ $\phi(0) = A\sin(k\,b)$

$\phi'(0) = A\,k\cos(k\,b)$

$\rightarrow \quad \delta\phi' = \cancel{2}\cancel{\alpha} k \cos(kb) \overset{!}{\underset{*}{=}} \frac{\cancel{2}m\mu}{\hbar^2}\cdot\cancel{\alpha}\sin(kb)$

$\tan(kb) \overset{!}{=} \frac{k\hbar^2}{m\mu}$

falls μ groß

$\rightsquigarrow kb$ klein

$\rightsquigarrow \tan(kb) \approx kb$

$\rightarrow \quad \boxed{b = \frac{\hbar^2}{m\mu}}$

R.B.: $\quad 0 \overset{!}{=} \alpha \sin(\underbrace{k(L+b)}_{\overset{!}{=}\pi\cdot n\ !}) = \phi^{\text{even}}(L)$

$\rightarrow \quad k_n^{\text{odd}} = \frac{\pi}{L+b}\cdot n \quad !$

$\rightarrow$ Eigenenergien symmetr. E. Fkten:

$E_n^{\text{even}} = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2}{2m}\cdot\frac{n^2}{(L+b)^2}$

$\rightarrow \quad 0 \overset{!}{=} \alpha \sin(\underbrace{kL}_{=\pi\cdot n}) = \phi^{\text{odd}}(L)$

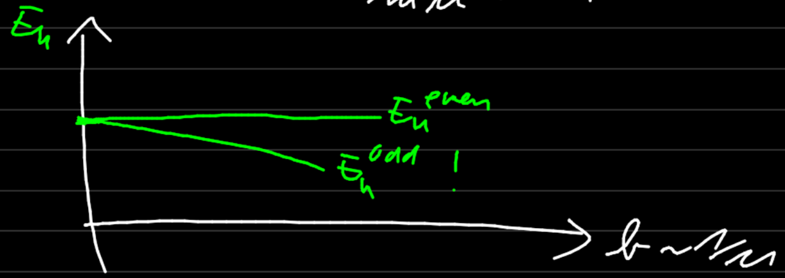
$\rightarrow \quad k_n^{\text{even}} = \frac{\pi}{L}\cdot n$

$$E_n^{odd} = \frac{\hbar^2 (k_n^{odd})^2}{2m} = \frac{\hbar^2\pi^2}{2m}\frac{n^2}{L^2}$$

---

für $\mu \to \infty$: $E_n^{odd} = E_n^{even}$

$\mu < \infty$: $\longrightarrow$ $b = \frac{\hbar^2}{m\mu} > 0$!



$\to$ Grundzust. energie :

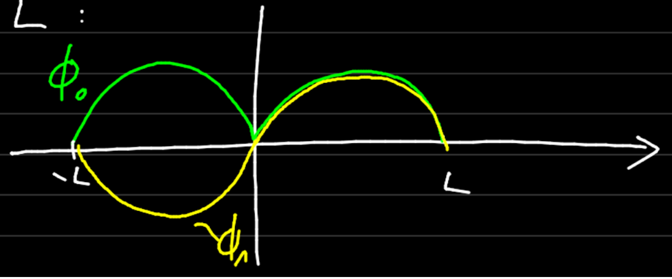
$$E_0 = E_1^{even} = \frac{\hbar^2\pi^2}{2m}\frac{1}{(L+b)^2}$$

$$E_1 = E_1^{odd} = \frac{\hbar^2\pi^2}{2m}\frac{1}{L^2}$$

$$\phi_0(x) = \tilde{c}\ \sin\left(k_1^{ev}(|x|+b)\right)$$

$$\phi_1(x) = c\ \sin(k_1^{od}\, x)$$

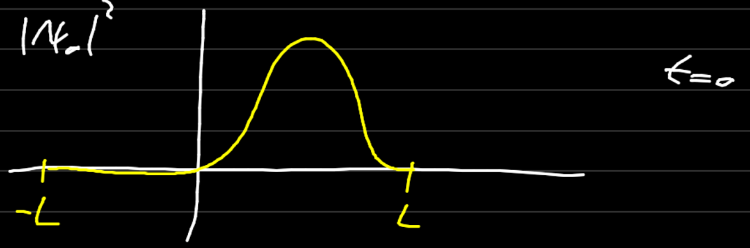
$0 < b \ll L$:



Dynamik: $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\phi_0\rangle + |\phi_1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$B = \left(|\phi_0\rangle, |\phi_1\rangle\right)$

$\Rightarrow |\psi_0|^2$



$E_0 = \frac{\hbar^2\pi^2}{2m}\cdot\frac{1}{(L+b)^2}$

$E_1 = \frac{\hbar^2\pi^2}{2m}\frac{1}{L^2}$ $\quad b \ll L$

$\Delta E = E_1 - E_0 = E_1\left(1 - \frac{1}{(1+b^2/L^2)}\right) \ll E_1$

$\hookrightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{iE_0 t/\hbar} \\ e^{iE_1 t/\hbar}\end{pmatrix} = \frac{e^{-i\Omega t}}{\sqrt{2}}\begin{pmatrix} e^{+i\frac{\omega t}{2}} \\ e^{-i\frac{\omega t}{2}}\end{pmatrix}$

$\Omega = \frac{E_0 + E_1}{2\hbar}$

$\omega = \frac{\Delta E}{\hbar} \ll \Omega\,!$

$\Rightarrow |\psi(t)|^2 = \dots$