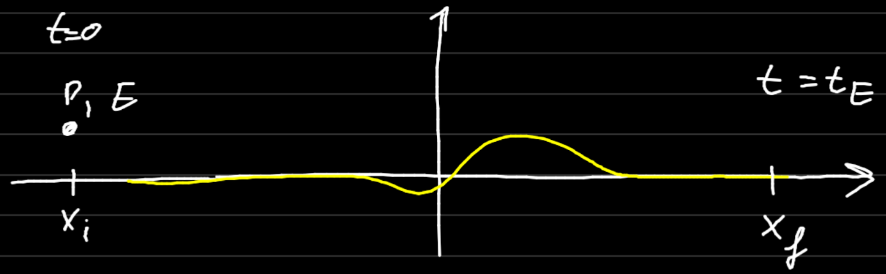
Nachtrag: Tunneleffekt im P.I. Form.

Teilchen, 1D, Potenzialschwelle $U(x)$



$E > U_{max}$: kl. Bahn x_{kl} von x_i nach x_f in Zeit t_E, Energie E

→

$$K(x_f, x_i, t_E) = d[x_{kl}]\, e^{i S[x_{kl}]/\hbar}$$

$$S[x_{kl}] = \int_0^{t_E} dt \Big(\underbrace{\frac{m}{2}\dot{x}(t)^2 - U(x(t)) + E}_{= \frac{m}{2}\dot{x}(t)^2 \;\leftarrow\; (\text{Energieerhaltung})} - E \Big)$$

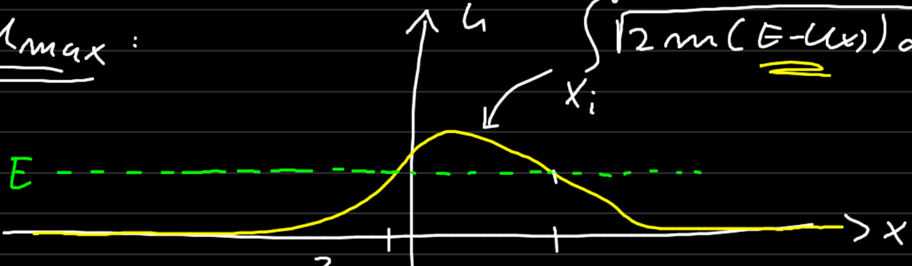
$$= -E t_E + \int_0^{t_E} dt\, \underbrace{m\dot{x}(t)}_{p(x(t)) = \sqrt{2m(E-U(x(t)))}}\, \dot{x}(t)$$

$$= -E t_E + \int_{x_i}^{x_f} \sqrt{2m(E-U(x))}\, dx$$

$$\rightsquigarrow K(x_f,x_i,t_E) = e^{-iEt_E}\, e^{\frac{i}{\hbar}\int_{x_i}^{x_f}\sqrt{2m(E-U(x))}\,dx}$$

<u>$E < U_{max}$:</u>

$$\int_{x_i}^{x_f}\sqrt{2m(E-U(x))}\,dx = i\int_{x_i}^{x_f}\sqrt{2m(U(x)-E)}\,dx$$



$$|\langle x_f|U_{t_E}|x_i\rangle|^2$$

$$\rightsquigarrow T \simeq |K(x_f,x_i,t_E)|^2 \simeq e^{-\frac{1}{\hbar}\int_{x_i}^{x_f}\sqrt{8m(U(x)-E)}\,dx} \qquad (*)$$

$\lambda \in \mathbb{C}$

$\lambda \in \mathbb{R}_+$

<u>genauer:</u>

$$K_\lambda(x_f,x_i,t) := K(x_f,x_i,\lambda t)$$

$$\hookrightarrow K(x_f,x_i,t) = K_\lambda(x_f,x_i,t/\lambda) =$$

$\lambda = i$!

$$K(x_f, x_i, t) = K_i(x_f, x_i, -it)$$

$$= \cdots = (\ldots)\, e^{-\frac{1}{\hbar}\int_0^t dt \left(\frac{m\dot{x}^2}{2} - \tilde{U}(x)\right)}$$

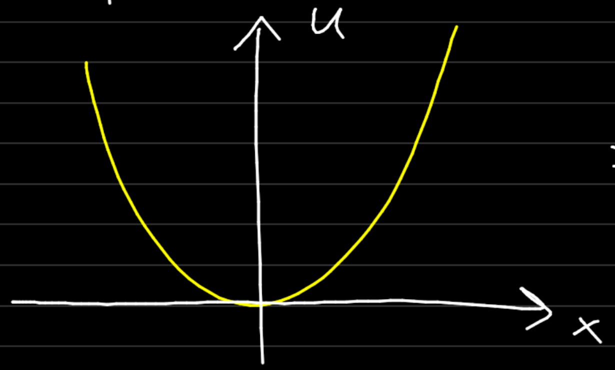
$$\tilde{U}(x) = -U(x)$$

$$\tilde{E} = -E$$

$\rightarrow$ (*)

Harmonischer Oszillator

Teilchen 1D, Masse m $\quad U(x) = \frac{m\omega^2}{2} x^2$



Frequenz ω

allg.: System mit Freiheitsgrad x harmon. Schwingungen um $x = 0$

1D H.O.: $H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2} x^2$

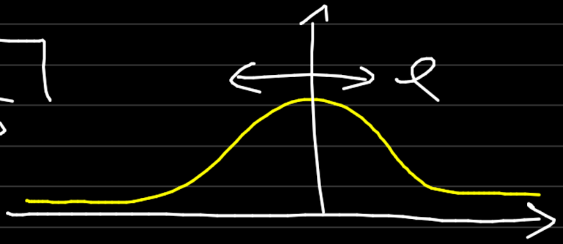
Eigenenergien

$$\boxed{E_n = \hbar\omega\left(n + \tfrac{1}{2}\right) \qquad n = 0, 1, 2, \dots}$$

Eigenfkt:

$$\boxed{\phi_0(x) = \frac{e^{-x^2/2\ell^2}}{(\pi \ell^2)^{1/4}}}$$

mit $\ell = \sqrt{\frac{\hbar}{m\omega}}$



$n > 0$:

$$\boxed{\phi_n(x) = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell} - \ell \frac{\partial}{\partial x}\right) \phi_{n-1}(x)}$$

analytisch: s.t. S.Gl.:

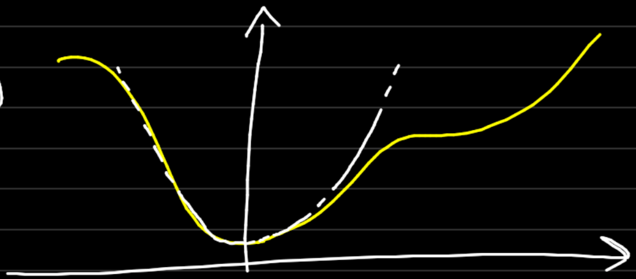
$$E\phi(x) = \left(-\frac{\hbar^2}{2m}\phi''(x) + \frac{m\omega^2}{2}\phi(x)\right)$$

konstruiere normierbare Eigenfkt

(. . . . . . .)

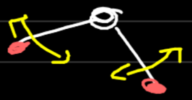
$\to$ nur möglich für diskr. Energien $E_n = \hbar\omega(n + \frac{1}{2})$

1)

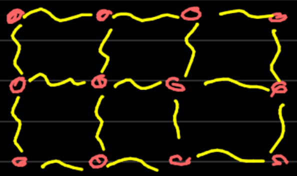


2) Molekülschwingungen:

H_2O



3)

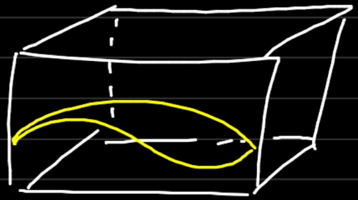


$\omega = c|\vec{k}|$

Gitterschwingungen, Wellenzahl $\vec{k}$

$\xrightarrow{q.m.}$ Phononen

4) el. mag. Feld:



$\vec{k}$

$\omega = c|\vec{k}|$

el. mag. Schwingungsmoden

↳ Photonen

$\hbar\omega$

Q. H.:

$E_\omega = n \Delta E$

⟶ Schwarzkörperstrahlung!

$\rightarrow \phi_n(x) = (\dots)\, e^{-x^2/2\ell^2}\, H_n(x/\ell)$

Hermite-Polynom ...

<u>algebraisch</u>: (Dirac, Born, ...)

(einfacher!)

Ausgangspunkt:

$$\boxed{a := \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\,\frac{\hat{p}}{m\omega}\right)} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\ell} + i\ell\,\frac{\hat{p}}{\hbar}\right)$$

$$\boxed{a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\,\frac{\hat{p}}{m\omega}\right)} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\ell} - i\ell\,\frac{\hat{p}}{\hbar}\right)$$

Kommutator:

$$\boxed{[a, a^+] = \mathbb{1}}$$

$$= \frac{\cancel{m\omega}}{2\hbar}\left(\underbrace{[\hat{x}, \hat{p}]}_{i\hbar}\left(\frac{-i}{\cancel{m\omega}}\right) + \underbrace{[\hat{p}, x]}_{-i\hbar}\frac{i}{\cancel{m\omega}}\right)$$

$$= \mathbb{1}$$

Darstellung von $\hat{x}, \hat{p}, \hat{H}$ durch a, a^+

$$\boxed{\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{m\omega}}\,(a^+ + a)\frac{1}{\sqrt{2}} \\ \hat{p} &= \sqrt{m\omega\hbar}\,(a^+ - a)\frac{i}{\sqrt{2}}\end{aligned}}$$

$$\hookrightarrow H = \frac{1}{2\cancel{m}}\left(\frac{-\hbar \cancel{m}\omega}{2}(a^+ - a)^2\right.$$

$$\left. + \frac{\cancel{m}\omega^{\cancel{2}}}{2}\left(\frac{\hbar}{2\cancel{m}\cancel{\omega}}(a^+ + a)^2\right)\right)$$

$$= \frac{\hbar\omega}{4}\left(a^+ a + a a^+\right)\cdot 2$$

$$= \frac{\hbar\omega}{2}\left(a^+ a + \underbrace{[a, a^+]}_{1} + a^+ a\right)$$

$$\boxed{H = \hbar\omega\left(a^+ a + \frac{1}{2}\right)}$$

$$(AB)^+ = B^+ A^+$$

mit $\hat{n} := a^+ a$

$$\to \boxed{H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)}$$

$$(\hat{n})^+ = (a^+ a)^+ = (a^+ (a^+)^+) = a^+ a = \hat{n}$$

Eigenwerte/-zust. von $\hat{n}$

$\hookrightarrow$ Eigenwerte/-zust. von $\underline{\underline{H}} = \hbar\omega(\hat{n} + \tfrac{1}{2})$

Eigenschaften von $\hat{n} = a^+ a$

1) $\hat{n} = \hat{n}^+$: $\hat{n}$ hermitesch

$\Rightarrow$ Eigenwerte <u>reell</u> !

2) $\hat{n}$ <u>positiv</u> $\longrightarrow$ Eigenwerte sind <u>semidefinit</u> <u>nicht</u> negativ

d.h. $\forall_\psi$ $\langle\psi|\hat{n}|\psi\rangle \geq 0$:

der Fall, da $\langle\psi|\hat{n}|\psi\rangle = \langle\psi|a^+ a|\psi\rangle$

$= \| a|\psi\rangle \|^2 \geq 0$

3) $\boxed{\begin{array}{l} \circ\ [\hat{n}, a] = -a \\ \circ\ [\hat{n}, a^+] = +a^+ \end{array}}$ ✓

$\circ\ [\hat{n}, a] = [a^+ a, a]$

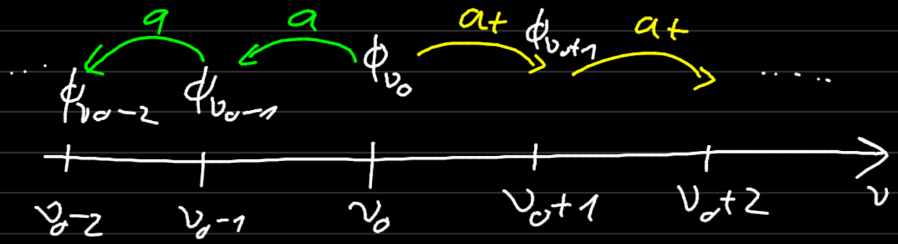
$= a^+ \underbrace{[a, a]}_{=0} + \overbrace{[a^+, a]}^{=-1} a = -a$

$\circ\ [\hat{n}, a^+] = a^+ \overbrace{[a, a^+]}^{+1} + \overbrace{[a^+, a^+]}^{=0} a = +a^+$

4) a) $|\phi_\nu\rangle$ normierter EZ zum EW $\nu > 0$

$$\Longrightarrow \begin{cases} \frac{1}{\sqrt{\nu+1}}\, a^+|\phi_\nu\rangle & \text{normierter EZ zum EW } \underline{\nu+1} \\ \frac{1}{\sqrt{\nu}}\, a|\phi_\nu\rangle & \text{norm. EZ zum EW } \underline{\underline{\nu-1}} \end{cases}$$

b) $a|\phi_0\rangle = 0$!



a, a^+ : Leiteroperatoren

$$\Gamma\ \hat{n}(a^+|\phi_\nu\rangle) = (\underbrace{[\hat{n}, a^+]}_{\hat{=}\, a^+} + a^+\underbrace{\hat{n})|\phi_\nu\rangle}_{=\nu}$$

$$= \underbrace{(\nu+1)}\, a^+|\phi_\nu\rangle$$

EV von $a^+|\phi_\nu\rangle$!

Norm: $\| a^+ |\phi_\nu\rangle \|^2$

$= \langle \phi_\nu | a a^+ | \phi_\nu \rangle$

$= \langle \phi_\nu | (\underbrace{a^+ a}_{\hat{n}} + \underbrace{[a, a^+]}_{= 1}) | \phi_\nu \rangle$

$= \nu + 1$

$\rightarrow \frac{a^+ |\phi_\nu\rangle}{\sqrt{\nu+1}} = |\phi_{\nu+1}\rangle$ norm. EZ. zum EW $\nu+1$

• $\hat{n}(a|\phi_\nu\rangle) = (\underbrace{[\hat{n}, a]}_{= -a} + a\hat{n}) |\phi_\nu\rangle$

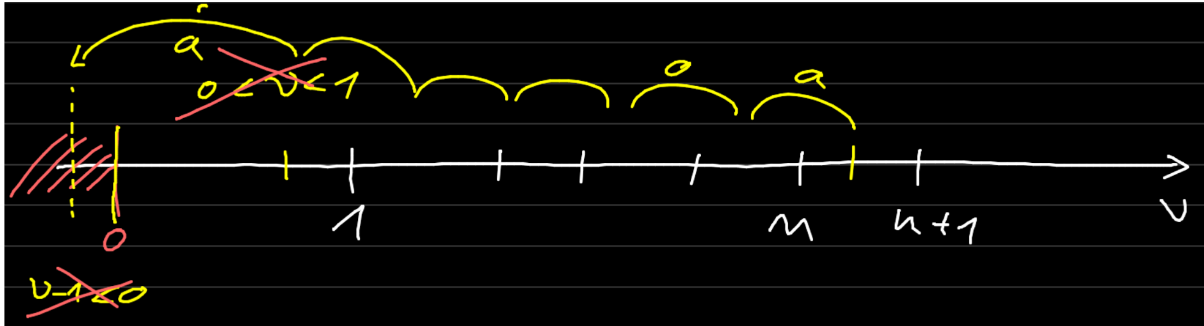
$= \underbrace{(\nu - 1)}_{EW} \underbrace{a|\phi_\nu\rangle}_{EZ}$

Norm: $\| a|\phi_\nu\rangle \|^2 = \langle \phi_\nu | \overbrace{a^+ a}^{\hat{n}} | \phi_\nu \rangle = \nu$

$\rightarrow \frac{a|\phi_\nu\rangle}{\sqrt{\nu}} = |\phi_{\nu-1}\rangle$ norm. EZ zum EW $\nu - 1$

l) $a|\phi_0\rangle$ ist von verschw. Norm $\rightarrow$ $\underline{\underline{a|\phi_0\rangle = 0}}$

$\| a|\phi_0\rangle \|^2 = \langle \phi_0 | \overbrace{a^+ a}^{\hat{n}} | \phi_0 \rangle = 0$!



es ex. kein EW $\nu \in \,]0,1[$!

$\rightarrow$ keine EWe in $]0,1[+ \mathbb{N}$!

d.h. $\boxed{\text{alle EWe von } \hat{n} \text{ in } \mathbb{N} \cup \{0\}\,!}$

$\rightarrow$ Spektrum von $\hat{n} = 0, 1, 2, 3, \dots$

$\rightarrow$ Spektrum von $H = \hbar\omega(\hat{n} + \frac{1}{2})$

ist $\boxed{E_n = \hbar\omega(n + 1/2) \qquad n \in \mathbb{N} \cup \{0\}}$

Eigenfunktion $\phi_0(x)$ von H zum EW $\frac{\hbar\omega}{2}$

= Eigenfkt. $\phi_0(x)$ von $\hat{n}$ zum EW 0:

$$a|\phi_0\rangle = 0 \iff \left(\hat{x} + i l \frac{\hat{p}}{\hbar}\right)'' \phi_0(x) \overset{!}{=} 0$$

d.h. $\left(\frac{x}{\ell}+\ell\frac{\partial}{\partial x}\right)\phi_0(x) \overset{!}{=} 0$

mit Lsg: $\boxed{\phi_0(x) = \frac{e^{-x^2/2\ell^2}}{(\pi\ell^2)^{1/4}}}$ ✓

4a) : $|\phi_{n-1}\rangle$ EZ. zum EW $n-1$

$\rightarrow \frac{a^\dagger|\phi_{n-1}\rangle}{\sqrt{n}} = |\phi_n\rangle$ norm. EZ zum EW n

$\rightarrow \boxed{\phi_n(x) = \frac{1}{\sqrt{2n}}\left(\frac{x}{\ell}-\ell\frac{\partial}{\partial x}\right)\phi_{n-1}(x)}$