Wdhlg.:

A Erhaltungsgröße $\iff [H, A] = 0$

d.h. $\langle A \rangle_{\psi(t)}$ konstant

$$\dot{\psi} = -\frac{i}{\hbar} H \psi$$

Übg.:

$$\frac{d}{dt} \langle A \rangle_{\psi(t)} = \left\langle \frac{i}{\hbar} [H, A] \right\rangle_{\psi(t)}$$

Hamilt. Mechanik

$$\frac{d}{dt} A(x(t)) = \{H, A\}_{x(t)}$$

$(q(t), p(t))$

Energiedarstellung:

$$H = \sum_{n=0}^{\infty} E_n |\varphi_n\rangle\langle\varphi_n|$$

Energieeigenw.

Energieeigenzust.

E

E_2

E_1 $|\varphi_1\rangle$

E_0 $|\varphi_0\rangle$

Zeitentw. op.:

$$e^{-iHt/\hbar} = U_t = \sum_{n=0}^{\infty} e^{\frac{-iE_n t}{\hbar}} |\varphi_n\rangle\langle\varphi_n|$$

$$\rightarrow |\psi(t)\rangle = U_t|\psi(0)\rangle = \sum_{n=0}^{\infty} e^{\frac{-iE_n t}{\hbar}} \langle\varphi_n|\psi_0\rangle|\varphi_n\rangle$$

$\rightarrow$ Energieeigenzust.e sind stationär:

$$|\psi(0)\rangle = |\varphi_n\rangle$$

$$\hookrightarrow |\psi(t)\rangle = e^{-iE_n t/\hbar} |\varphi_n\rangle$$

$$\rightarrow \langle A\rangle_{\psi(t)} = \langle A\rangle_{\varphi_n} \quad \text{konstant!}$$

- $|\psi(0)\rangle = \alpha|\varphi_n\rangle + \beta|\varphi_m\rangle$

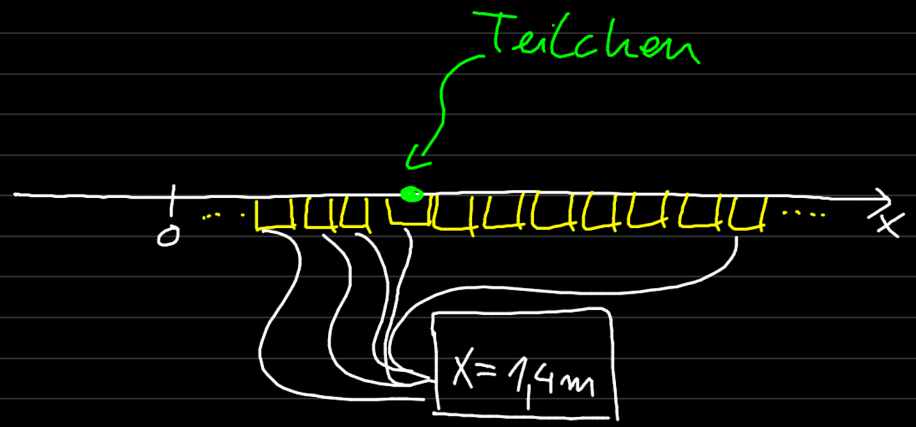
$$\hookrightarrow |\psi(t)\rangle = \alpha e^{-iE_n t/\hbar}|\varphi_n\rangle + \beta e^{-iE_m t/\hbar}|\varphi_m\rangle$$

$$\rightsquigarrow \langle A\rangle_t = a + b\cos\left(\frac{\Delta E_{nm}}{\hbar}t + \vartheta\right)$$

$$\omega_{nm} = \frac{\Delta E_{nm}}{\hbar} = \frac{E_n - E_m}{\hbar}$$

$(\hbar = 1)$

Quantenmechanik des Punktteilchens (10)



Ortsmessung

→ Koordinate $x \in \mathbb{R}$

Idealisierung

Ortsobservable x $\hat{=}$ herm. Oper. $\hat{X}$ mit ("Ortsoperator")

Eigenwerten $\{x\}_{x\in\mathbb{R}}$ (*)

zu Eigenzuständen $\{|\varphi_x\rangle\}_{x\in\mathbb{R}}$ (*)

(*) Kontinuum an EWen/EZen!

Mathemat.:

Lin. Alg. ($\dim\mathcal{H}<\infty$) ⟶ Funktionalanaly. ($\dim\mathcal{H}=\infty$)

hier: Anpassung des Dirac-Formal.
für kontinuierlichen Spektrums.

diskret

herm. Op. A $\hat{=}$ Obs.

$\to$ EWe $\{a_1, a_2, \ldots\ldots\} = \{a_i\}_{i=1,2,\ldots}$

EZe $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots\} = \{|\varphi_i\rangle\}_{i=1,2,\ldots}$

$A|\varphi_i\rangle = a_i|\varphi_i\rangle$

ONB

kontinuierlich

Ortsop. $\hat{x}$

EWe $\{x\}_{x\in\mathbb{R}}$

EZe $\{|\varphi_x\rangle\}_{x\in\mathbb{R}}$

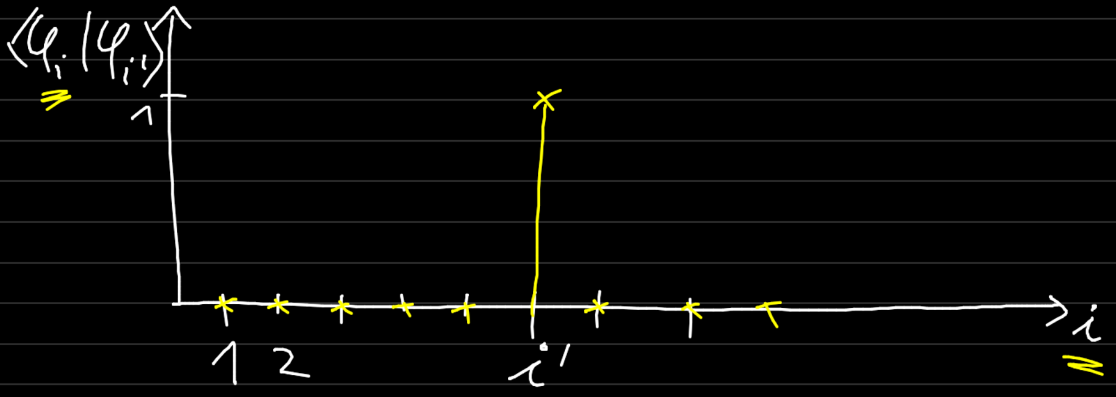
$\hat{x}|\varphi_x\rangle = x|\varphi_x\rangle$

Op. reelle Zahl.

Orthonormalität

diskret

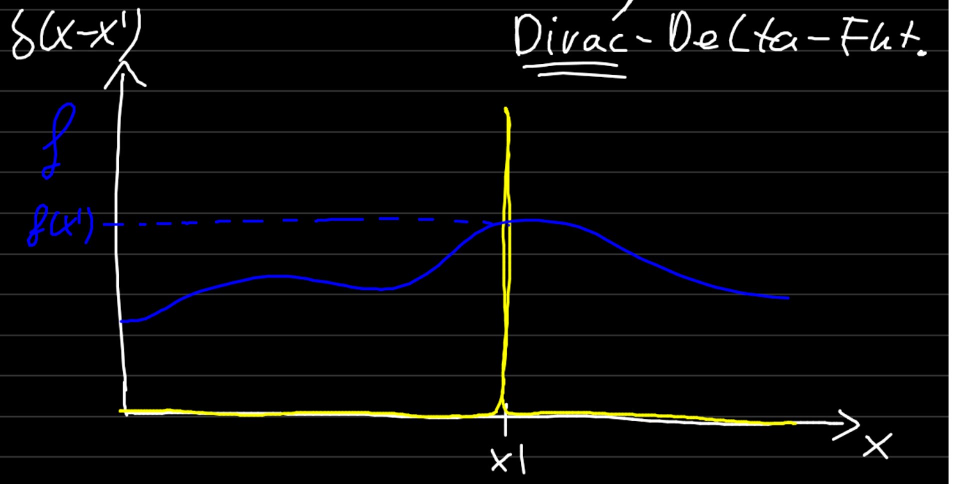
$$\langle \varphi_i | \varphi_{i'} \rangle = \delta_{ii'}$$



kontinuierlich

$$\boxed{\langle \varphi_x | \varphi_{x'} \rangle = \delta(x-x')}$$

Dirac-Delta-Fkt.



diskret

- $\delta_{ii'} = 0 \qquad i \neq i'$
- $\sum_{i=1}^{\infty} \delta_{ii'} = 1$
- $\sum_{i=1}^{\infty} f_i \, \delta_{ii'} = f_{i'}$

kontin.

- $\delta(x-x') = 0 \qquad x \neq x'$
- $\int_{-\infty}^{\infty} dx \, \delta(x-x') = 1$
- $\int_{-\infty}^{\infty} dx \, f(x) \, \delta(x-x') = \underline{f(x')}$

definierenden Eigenschaften

$\rightarrow$ Rechenregeln!

diskret

Vollständigkeit

kontin.

der Ete $\{|\varphi_i\rangle\}_{i=1,2,\dots}$

$$\mathbb{1}_{\mathcal{H}} = \sum_i |\varphi_i\rangle\langle\varphi_i|$$

$$|\psi\rangle = \mathbb{1}_{\mathcal{H}}|\psi\rangle = \sum_i |\varphi_i\rangle\underbrace{\langle\varphi_i|\psi\rangle}_{\psi_i}$$

$$= \sum \psi_i |\varphi_i\rangle$$

$\{|\varphi_x\rangle\}_{x\in\mathbb{R}}$

$$\boxed{\mathbb{1}_{\mathcal{H}} \overset{!}{=} \int_{-\infty}^{+\infty} dx\, |\varphi_x\rangle\langle\varphi_x|}$$

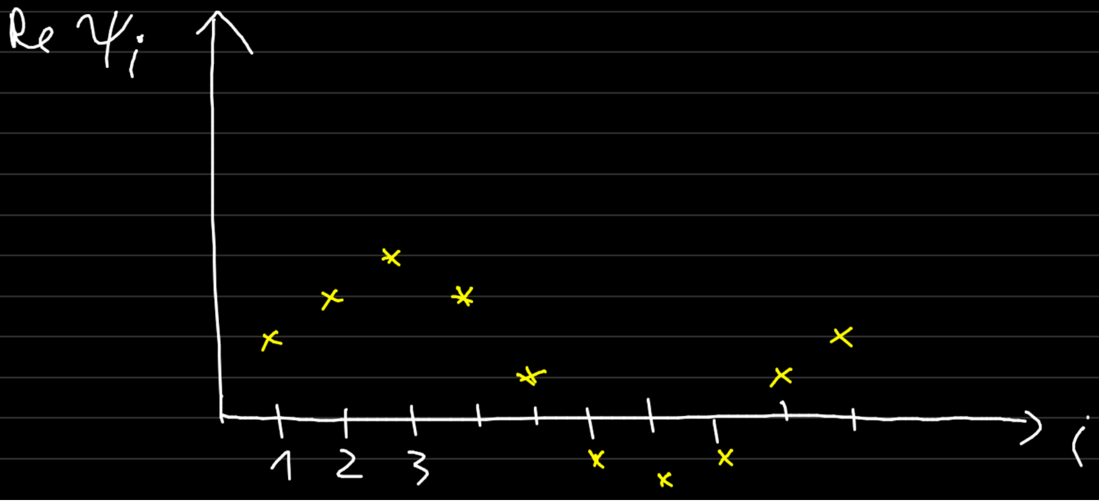
$$|\psi\rangle = \mathbb{1}_{\mathcal{H}}|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx\, |\varphi_x\rangle\overbrace{\langle\varphi_x|\psi\rangle}^{\psi(x)}$$

diskret

$$|\psi\rangle = \sum \psi_i \, |\varphi_i\rangle$$

i.-Komp. vom $|\psi\rangle$ bzg $\{|\varphi_i\rangle\}$

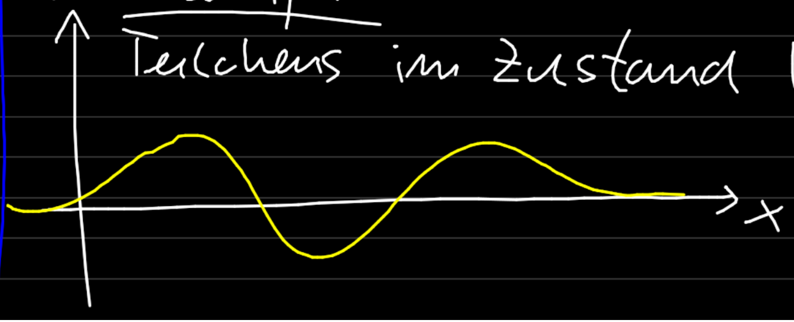


konti.

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \; \psi(x) \, |\varphi_x\rangle$$

mit $\boxed{\psi(x) := \langle \varphi_x | \psi \rangle \in \mathbb{C}}$

Wellenfkt. am Ort x des Teilchens im Zustand $|\psi\rangle$



Skalarprodukt

diskret

$$\langle\psi|\chi\rangle = \langle\psi|\mathbb{1}_{\mathcal{H}}|\chi\rangle$$

$$= \langle\psi|\Big(\sum_i |\varphi_i\rangle\langle\varphi_i|\Big)|\chi\rangle$$

$$= \sum_i \underbrace{\langle\psi|\varphi_i\rangle}_{\psi_i^*}\underbrace{\langle\varphi_i|\chi\rangle}_{\chi_i}$$

$$\langle\psi|\chi\rangle = \sum_i \psi_i^* \chi_i \quad \checkmark$$

konti

$$\langle\psi|\chi\rangle = \langle\psi|\mathbb{1}_{\mathcal{H}}|\chi\rangle$$

$$= \langle\psi|\Big(\int_{-\infty}^{\infty} dx\, |\varphi_x\rangle\langle\varphi_x|\Big)|\chi\rangle$$

$$= \int_{-\infty}^{\infty} dx\, \underbrace{\langle\psi|\varphi_x\rangle}_{\psi^*(x)}\underbrace{\langle\varphi_x|\chi\rangle}_{\chi(x)}$$

$$\boxed{\langle\psi|\chi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\,\chi(x)}$$

Norm

diskret

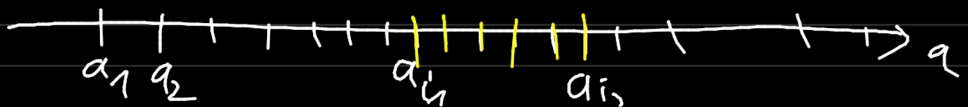
$$\| |\psi\rangle \|^2 = \langle\psi|\psi\rangle = \sum_{i=1}^{\infty} |\psi_i|^2$$

kont.

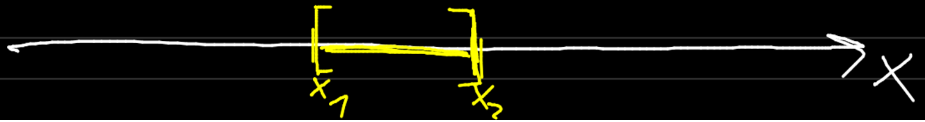
$$\| |\psi\rangle \|^2 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx \, |\psi(x)|^2$$

Bornsche Regel

p = Wkt., das Messwert a von Obs A im $\{a_{i_1}, \dots, a_{i_n}\}$, System im Zus $|\psi\rangle$



p = Wkt, das Ortsmessung ergibt $x \in [x_1, x_2]$



diskret

$$P = \sum_{i=i_1}^{i_2} p_i = \sum_{i=i_1}^{i_2} |\langle \varphi_i | \psi \rangle|^2 = \sum_i |\psi_i|^2$$

konti.

$$P = \int_{x_1}^{x_2} dx \, |\langle \varphi_x | \psi \rangle|^2 = \int_{x_1}^{x_2} dx \, |\psi(x)|^2$$

$$\boxed{\varrho(x) := |\langle \varphi_x | \psi \rangle|^2 = |\psi(x)|^2}$$

Aufenthaltswahrscheinlichkeits-
dichte des Teilchens (im Zust $|\psi\rangle$)
am Ort x
(Born 1926)

diskret:

Erwartungswerte

$$\langle A \rangle_{|\psi\rangle} = \sum_i a_i p_i = \sum_i a_i |\langle\varphi_i|\psi\rangle|^2 =$$

alt

$$\langle A \rangle_{|\psi\rangle} = \langle\psi|A|\psi\rangle = \overset{!}{=}$$

$$A = \sum a_i |\varphi_i\rangle\langle\varphi_i|$$

$$\boxed{\langle x \rangle_{|\psi\rangle}} = \int_{-\infty}^{\infty} dx \; x \, \rho(x)$$

$$= \int_{-\infty}^{\infty} dx \; x \, |\psi(x)|^2$$

$$\boxed{\langle\psi|\hat{x}|\psi\rangle} = \langle\psi|\left(\int dx \; x |\varphi_x\rangle\langle\varphi_x|\right)|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx \; x \underbrace{\langle\psi|\varphi_x\rangle\langle\varphi_x|\psi\rangle}_{|\psi(x)|^2}$$

$$\hat{x}\, \mathbb{1}_{\mathcal{H}} = \hat{x} = \int_{-\infty}^{\infty} dx \; x |\varphi_x\rangle\langle\varphi_x|$$

$$A = \sum_i a_i |\varphi_i\rangle\langle\varphi_i|$$

$$f(a) = \cdots\cdots$$

$$f(A) = \sum_i f(a_i)\, |\varphi_i\rangle\langle\varphi_i|$$

$$A^n = \sum_i a_i^n\, |\varphi_i\rangle\langle\varphi_i|$$

$$e^A = \sum_i e^{a_i}\, |\varphi_i\rangle\langle\varphi_i|$$

Spektraldarst.

$$\hat{x} = \int_{-\infty}^{\infty} dx\; x\, |\varphi_x\rangle\langle\varphi_x|$$

$$f(x) = \cdots$$

$$f(\hat{x}) = \int_{-\infty}^{\infty} dx\; f(x)\, |\varphi_x\rangle\langle\varphi_x|$$

$$f(x) = e^{ikx}, \quad f(x) = x^2, \quad \cdots\cdots$$

"Ortsdarstellung" von $\hat{x}$:

$$\begin{tikzcd}
{|\psi\rangle} \arrow[r, "\hat{x}\cdot"] \arrow[d, "\{|\varphi_x\rangle\}"'] & {|\tilde{\psi}\rangle := \hat{x}|\psi\rangle} \arrow[d] \\
\psi(x) \arrow[r, "x\cdot"] & \tilde{\psi}(x) = x\,\psi(x)
\end{tikzcd}$$

$$\tilde{\psi}(x) = \langle\varphi_x|\tilde{\psi}\rangle = \langle\varphi_x|\hat{x}|\psi\rangle$$

$$= \langle\tilde{\psi}|\hat{x}|\varphi_x\rangle^* = \langle\tilde{\psi}|\underset{\in\mathbb{R}}{x}|\varphi_x\rangle^* =$$

$$\rightarrow\ = x\,\langle\psi|\varphi_x\rangle^* = x\,\underbrace{\langle\varphi_x|\psi\rangle}_{\psi(x)}$$

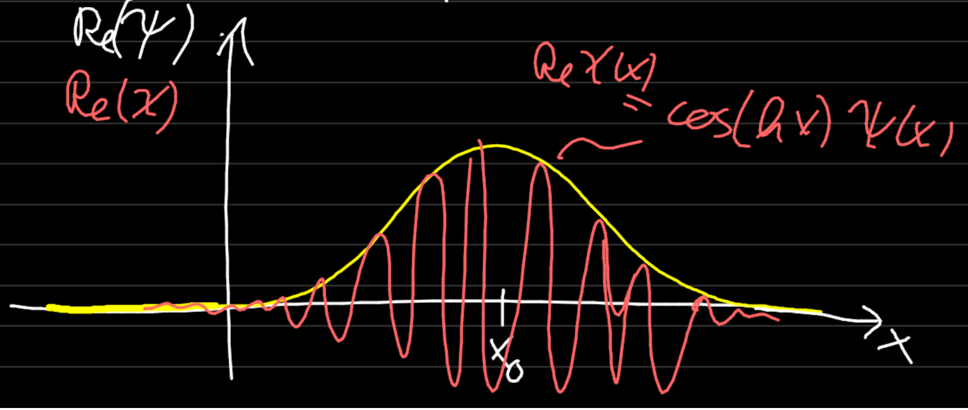
analog:

$$\begin{tikzcd}
{|\psi\rangle} \arrow[r, "f(\hat{x})"] \arrow[d] & {|\tilde{\psi}\rangle = f(\hat{x})|\psi\rangle} \arrow[d] \\
\psi(x) \arrow[r, "f(x)"] & \tilde{\psi}(x) = f(x)\,\psi(x)
\end{tikzcd}$$

Beispiel:

1) Teilchen im Zust $|\psi\rangle$ mit Wellenfkt.

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\, e^{-\frac{(x-x_0)^2}{4\sigma^2}}$$



$$\| |\psi\rangle \| = \int_{-\infty}^{\infty} dx\, |\psi(x)|^2$$

$$= \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-x_0)^2}{2\sigma^2}} = 1 \checkmark$$

$$\langle x \rangle_{|\psi\rangle} = \int_{-\infty}^{\infty} dx\, x\, |\psi(x)|^2 \overset{!}{=} x_0 \checkmark$$

2) Teilchen im Zust. χ mit <u>Wellenfkt.</u> $\chi(x) = e^{ikx} \cdot \psi(x)$

$$\langle \psi | \chi \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \, \chi(x)$$

$$= \int_{-\infty}^{\infty} dx \; \frac{e^{-\frac{(x-x_0)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \cdot e^{ikx}$$

$$\underset{x \to x + x_0}{=} \underbrace{\left(\int_{-\infty}^{\infty} dx \; \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \, e^{ikx} \right)}_{= e^{-\frac{k^2\sigma^2}{2}}} \cdot e^{ikx_0}$$

$b = ik$

$$\int e^{-ax^2 + bx} \, dx = \sqrt{\frac{\pi}{a}} \, e^{b^2/4a}$$