Lösungshinweise Blatt 4

11) "$\Leftarrow$": $[H,p]=0 \Rightarrow [\underbrace{e^{-iHt}}_{= U(t)}, \underbrace{e^{-ips}}_{= T(s)}] = 0.$ $(\hbar=1)$

"$\Rightarrow$": $\forall t,s: \; [U(t), U(s)] = 0$

$$\Rightarrow \quad 0 = -\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 [U(t), U(s)]$$

$$= \Big[\underbrace{i\frac{d}{dt}U(t)\Big|_0}_{= H}, \underbrace{i\frac{d}{ds}U(s)\Big|_0}_{= p}\Big] .$$

12a) <u>$n=1$:</u> $[A,B] = 1\cdot \underbrace{A^0}_{= 1} [A,B]$ ✓

<u>$n \to n+1$:</u>

$$[A^{n+1}, B] = A \underbrace{[A^n, B]}_{\overset{\text{I.V.}}{=} n A^{n-1}[A,B]} + [A,B]A^n$$

$$\overset{[[A,B],A]=0}{=} (n+1) A^{(n+1)-1} [A,B] \quad ✓$$

12b) $$[f(A), B] = \sum_n c_n [A^n, B] \overset{a)}{=} \underbrace{\sum_n c_n n A^{n-1}}_{\hat{=} f'(A)} [A,B] .$$

12.1)

mit $[x,p] = i\hbar$ und ℓ_1 folgt

- $\frac{i}{\hbar}[p^2,x] = \frac{2i}{\hbar}\, p\,[p,x] = 2p$
- $\frac{i}{\hbar}[x^2,p] = \frac{2i}{\hbar}\, x\,[x,p] = -2x$
- $\frac{i}{\hbar}[U(x),p] = \frac{i}{\hbar}\, U'(x)\,[x,p] = -U'(x)$
- $[e^{iax},p] = ia\,e^{iax}\,[x,p] = -\hbar a\,e^{iax}$
- $[e^{ibp},x] = ib\,e^{ibp}\,[p,x] = \hbar b\,e^{ibp}$

13a)

$$H = \frac{p^2}{2m} + U(x) \xrightarrow{12c)} \frac{i}{\hbar}[H,x] = p/m$$

$$\frac{i}{\hbar}[H,p] = -U'(x)$$

$$\rightarrow \frac{d}{dt}\langle x\rangle_{\psi_t} = \left\langle \frac{i}{\hbar}[H,x]\right\rangle_{\psi_t} = \frac{1}{m}\langle p\rangle_{\psi_t}$$

$$\frac{d}{dt}\langle p\rangle_{\psi_t} = \left\langle \frac{i}{\hbar}[H,p]\right\rangle_{\psi_t} = -\langle U'(x)\rangle_{\psi_t} \quad .$$

13 b) $\langle x \rangle_{\psi_t}$ und $\langle p \rangle_{\psi_t}$ genügen exakt den kl. Bewegungsgl.en g.d.w. $\langle U'(x) \rangle_{\psi(t)} \overset{!}{=} U'(\langle x \rangle_{\psi_t})$;

den Fall für $U'(x)$ linear!

$\rightarrow$ für rationales Pot. $U(x)$ vom Grad ≤ 2 folgen q.m. EWe exakt den kl. B.G.en!

insbesondere für harm. Oszillator:

$$U(x) = \frac{m\omega^2}{2} x^2$$

d.h. $$\langle x \rangle_{\psi_t} \overset{!}{=} \langle x \rangle_{\psi_0} \cos \omega t + \frac{\langle p \rangle_{\psi_0}}{m\omega} \sin \omega t .$$

14 a) Mit $H = p^2/2m$ ist $|\tilde{e}_{\hbar k}\rangle$ insb. Energie-eigenzust. zur Energie $E_k = \hbar^2 k^2/2m$

$$\rightarrow |\psi(t)\rangle = U(t)\,|\psi(0)\rangle = U(t) \int \frac{dk}{2\pi}\, \tilde{\psi}(k)\, |\tilde{e}_{\hbar k}\rangle$$

$$= \int \frac{dk}{2\pi}\, \tilde{\psi}(k)\, U(t)\, |\tilde{e}_{\hbar k}\rangle = \int \frac{dk}{2\pi}\, \underbrace{\tilde{\psi}(k)\, e^{-i\frac{\hbar k^2}{2m}t}}_{\hat{=}\ \tilde{\psi}(k,t)}\, |\tilde{e}_{\hbar k}\rangle$$

14b)

$$\tilde\psi(k) = \int dx\, e^{-\overbrace{ik}^{\tilde b}x}\,\frac{e^{-\overbrace{x^2/4\sigma^2}^{\tilde a}}}{(2\pi\sigma^2)^{1/4}} = \sqrt{2}\,(2\pi\sigma^2)^{1/4}\, e^{-\sigma^2 k^2}$$

$$\rightarrow \quad \tilde\psi(k,t) = \sqrt{2}\,(2\pi\sigma^2)^{1/4}\, e^{-i\frac{\hbar t}{2m}k^2 - \sigma^2 k^2}$$

$$\rightarrow \quad \psi(x,t) = \int \frac{dk}{2\pi}\,\tilde\psi(k,t)\, e^{+ikx}$$

$$= \frac{(2\pi\sigma^2)^{1/4}}{\sqrt{2}\,\pi}\int dk\; e^{-\underbrace{\left(\sigma^2 + i\frac{\hbar t}{2m}\right)}_{a}k^2 + \underbrace{ix}_{b}\,k}$$

$$= \frac{(2\pi\sigma^2)^{1/4}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{|a|}}\; e^{-x^2/4a}$$

mit $|a| = \sqrt{\sigma^4 + \left(\frac{\hbar t}{2m}\right)^2} = \overbrace{\sigma^2\left(1 + t^2/\tau^2\right)^{1/2}}^{\overset{(*)}{=}\;\sigma\cdot\sigma_t}$

und $\frac{1}{a} = \frac{a^*}{|a|^2} = \frac{\sigma^2 - i\,\sigma^2 t/\tau}{\sigma^4\left(1 + t^2/\tau^2\right)} \underset{(*)}{=} \frac{1}{\sigma_t^2} - i\,\frac{t}{\sigma_t^2\tau}$

ergibt sich

$$\psi(x,t) = \frac{1}{(2\pi\sigma_t^2)^{1/4}}\; e^{-\frac{x^2}{4\sigma_t^2}}\; e^{+i\frac{x^2}{4\sigma_t^2}\cdot\frac{t}{\tau}}$$

mit $\tau = \frac{2m\sigma^2}{\hbar}$; $\sigma_t = \sigma\cdot\left(1 + t^2/\tau^2\right)^{1/2}$ (*)