Lösungshinweise Blatt 5

16a) $\psi_{1/2}(x)$ genügen S.Gl. $\psi''_{1/2}(x) = \underbrace{\frac{2m}{\hbar^2}(U(x)-E)}_{=\alpha}\psi_{1/2}(x)$

$\rightarrow \quad (\psi_1'\psi_2 - \psi_1\psi_2')' = \psi_1''\psi_2 - \psi_1\psi_2'' = \alpha(\psi_1\psi_2 - \psi_2\psi_1) = 0\ !$

d.h. $\psi_1'\psi_2 - \psi_1\psi_2' = \text{konst.} = c$; da $\psi_{1/2}(x)$ und $\psi_{1/2}'(x)$ für $x \to \infty$ verschwinden ($\psi_{1/2}$ normierbar!)

ist $c = 0$ und somit $\frac{\psi_1'}{\psi_1} - \frac{\psi_2'}{\psi_2} = 0$

d.h. $(\ln\psi_1 - \ln\psi_2)' = 0$, also $\ln\psi_1 = \ln\psi_2$,

d.h. ψ_1 proportional ψ_2.

16b) $\frac{im}{\hbar}[H, x] = \frac{im}{\hbar}\left[\frac{p^2}{2m} + U(x), x\right] = p$. $\quad [p,x] = -i\hbar$

$\rightarrow \quad \langle p \rangle_{\psi_E} = \frac{im}{\hbar}\langle \psi_E | Hx - xH | \psi_E \rangle =$

$= \frac{im}{\hbar}\langle \psi_E | Ex - xE | \psi_E \rangle = 0.$

17)

Für $x \neq 0$ und $E<0$ lautet S.Gl.: $\psi_E''(x) = -\frac{2mE}{\hbar^2}\psi_E(x)$

$\rightarrow$ Lsgen $e^{\pm \varkappa_E x}$ mit $\varkappa_E = (-2mE/\hbar^2)^{1/2}$

$\rightarrow$ stetige, normierbare Lsg. $\psi_E(x) = c\, e^{-\varkappa_E |x|}$;

Integration der S.G.l. $\psi_E''(x) = \frac{2m}{\hbar^2}(-u\,\delta(x) - E)\,\psi(x)$

über $[-\varepsilon, \varepsilon]$ ergibt für $\varepsilon \to 0$ Anschlussbedingung

$$\underbrace{\psi_E'(0+) - \psi_E'(0-)}_{\substack{\parallel \\ -2\varkappa_E\,\psi(0)}} \overset{!}{=} -\frac{2mu}{\hbar^2}\,\psi(0) \qquad \text{(vgl. Vrlsg.)}$$

d.h. $\quad -2mE/\hbar^2 = \varkappa_E^2 \overset{!}{=} \left(\frac{mu}{\hbar^2}\right)^2$

$\rightarrow \quad E = -mu^2/2\hbar^2$

$$\psi_E(x) = \varkappa_E\, e^{-\varkappa_E |x|} .$$

18 a)

$X_{11} = X_{22} = a/2 \qquad$ (Symmetrie!)

$$X_{12} = \frac{2}{a}\int_0^a x \sin\frac{\pi}{a}x \,\sin\frac{2\pi}{a}x\, dx$$

$$= \frac{2a}{\pi^2}\int_0^\pi y \sin y \,\sin 2y\, dy = -\left(\frac{4}{3\pi}\right)^2 a$$

$\uparrow$
$x = \frac{a}{\pi}y$

$X_{21} = X_{12}$

$$\rightarrow \quad \underline{\underline{X}} = \frac{a}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \left(\frac{4}{3\pi}\right)^2 a \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$p_{11} = p_{22} = 0 \qquad (\text{nach } 16\, b))$$

$$p_{21} = \frac{2}{a} \int_0^a \sin\frac{2\pi}{a}x \left(-i\hbar \frac{\partial}{\partial x}\right) \sin\frac{\pi}{a}x \, dx$$

$$= -i\hbar \frac{2\pi}{a^2} \int_0^a \sin\frac{2\pi}{a}x \, \cos\frac{\pi}{a}x \, dx$$

$$\underset{x = a y/\pi}{=} \; -i\hbar \frac{2}{a} \int_0^\pi \cos y \, \sin 2y = -i \frac{8}{3} \frac{\hbar}{a}$$

$$p_{12} = p_{12}^* \qquad \longrightarrow \qquad \underline{\underline{p}} = \frac{8}{3}\frac{\hbar}{a} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

- $H_{ij} = E_i \delta_{ij}$:

$$\underline{\underline{H}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

18 b)

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\rightarrow \quad \psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-iE_1 t/\hbar} \\ e^{-iE_2 t/\hbar} \end{pmatrix} = \frac{e^{-iE_2 t/\hbar}}{\sqrt{2}} \begin{pmatrix} e^{i\Omega t} \\ 1 \end{pmatrix}$$

mit $\Omega = (E_2 - E_1)/\hbar$

$$\rightarrow \quad \langle X \rangle_{\psi(t)} = \psi(t)^{+} \, \underline{\underline{X}} \, \psi(t)$$

$$\overset{a)}{=} \psi^{+}(t) \left(\frac{a}{2} \mathbb{1} - \left(\frac{4}{3\pi}\right)^{2} a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \psi(t)$$

$$= \frac{a}{2} - \frac{4}{3\pi} a \left(\frac{e^{i\Omega t} + e^{-i\Omega t}}{2} \right)$$

$$= \frac{a}{2} - \frac{4a}{3\pi} \cos \Omega t$$

$$\langle P \rangle_{\psi(t)} = \psi^{+}(t) \, \underline{\underline{P}} \, \psi(t)$$

$$= \frac{8\hbar}{3a} \psi^{+}(t) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \psi(t)$$

$$= \frac{8\hbar}{3a} \left(\frac{i e^{-i\Omega t} - i e^{+i\Omega t}}{2} \right)$$

$$= \frac{8\hbar}{3a} \sin \Omega t$$

$$\langle H \rangle_{\psi(t)} = \langle H \rangle_{\psi(0)} = \psi^{+}(0) \, \underline{\underline{H}} \, \psi(0)$$

$$= \frac{1}{2} (E_1 + E_2)$$

18 c) $P_n(0) = |\langle \psi_n | \chi \rangle|^2 = \left| \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{\pi n}{a} x\right) dx \right|^2$

$$= 2 \left| \frac{1}{a} \frac{a}{\pi n} \underbrace{\cos \frac{\pi n x}{a} \Big|_a^0}_{\substack{\shortparallel \\ 1 - \cos \pi n}} \right|^2$$

$1 - \cos \pi n \begin{cases} 2 : n \text{ ungerade} \\ 0 : n \text{ gerade} \end{cases}$

$\rightarrow P_n(0) = \frac{8}{\pi^2} \cdot \frac{1}{n^2}$ falls n ungerade

sonst $P_n(0) = 0$.

$$P_n(t) = |\langle \psi_n | e^{-iHt/\hbar} | \chi \rangle|^2$$

$$= | e^{iE_n t/\hbar} \langle \psi_n | \chi \rangle |^2 = P_n(0)$$