Lösungshinweise Blatt 7

24 a) Die Impulswellenfkt. des Zustands $\hat{x}|\psi\rangle$ ist

$$\langle \tilde{\varphi}_k | \hat{x} | \psi \rangle = \int dx \, \langle \tilde{\varphi}_k | \hat{x} | x \rangle \langle x | \psi \rangle$$

$$= \int dx \; x \, e^{-ikx} \, \psi(x)$$

$$= i \frac{\partial}{\partial k} \int dx \; e^{-ikx} \psi(x) = i \frac{\partial}{\partial k} \tilde{\psi}(k) .$$

b)

$$\langle x \rangle_{|\psi\rangle} = \int \frac{dk}{2\pi} \underbrace{\langle \psi | \tilde{\varphi}_k \rangle}_{=} \underbrace{\langle \tilde{\varphi}_k | x | \psi \rangle}_{=}$$

$$= \int \frac{dk}{2\pi} \; \tilde{\psi}^*(k) \; i \frac{\partial}{\partial k} \tilde{\psi}(k) ,$$

$$\langle p \rangle_{|\psi\rangle} = \int \frac{dk}{2\pi} \langle \psi | \underbrace{p | \tilde{\varphi}_k \rangle}_{\hbar k |\tilde{\varphi}_k\rangle} \langle \tilde{\varphi}_k | \psi \rangle$$

$$= \int \frac{dk}{2\pi} \; \tilde{\psi}^*(k) \; \hbar k \; \tilde{\psi}(k) .$$

c) z.z.: $e^{i p_0 \hat{x}} | \tilde{\varphi}_p \rangle \stackrel{!}{=} | \tilde{\varphi}_{p+p_0} \rangle$; $(\hbar = 1)$

$$e^{i p_0 \hat{x}} | \tilde{\varphi}_p \rangle = \int dx \; e^{i p_0 \hat{x}} |x\rangle\langle x | \tilde{\varphi}_p \rangle$$

$$= \int dx \; |x\rangle \underbrace{e^{i p_0 x} \, e^{ipx}}_{= \langle x | \tilde{\varphi}_{p+p_0} \rangle} = \overbrace{\int dx \, |x\rangle\langle x|}^{= 1} \tilde{\varphi}_{p+p_0} \rangle .$$

d) $\tilde{T}^{+}(p_0) = \left(e^{ip_0\hat{x}}\right)^{+} \underset{\underset{\hat{x} \text{ hermitesch}}{\nearrow}}{=} e^{-ip_0\hat{x}} = \tilde{T}(p_0)^{-1}$,

$$[\hat{p}, \tilde{T}(p_0)]\,|\varphi_p\rangle = (\hat{p}\,\tilde{T}(p_0) - \tilde{T}(p_0)\,\hat{p})\,|\varphi_p\rangle$$

$$= (p + p_0 - p)\,|\varphi_{p+p_0}\rangle$$

$$= p_0\,\tilde{T}(p_0)\,|\varphi_p\rangle \quad .$$

e)

$$\langle p \rangle_{\tilde{T}_{p_0}|\psi\rangle} = \langle\psi|\,\tilde{T}_{p_0}^{+}\,\hat{p}\,\tilde{T}_{p_0}\,|\psi\rangle$$

$$\overset{d)}{=} \langle\psi|\,\tilde{T}_{p}^{+}\,(\tilde{T}_{p_0}\,\hat{p} + p_0\,\tilde{T}_{p_0})\,\psi\rangle$$

$$= \langle\psi|\hat{p}|\psi\rangle + p_0\langle\psi|\psi\rangle = \langle p\rangle_{|\psi\rangle} + p_0 \quad .$$

f) $\tilde{T}_{\hbar k_0}\,|\psi\rangle = \int dx\, \tilde{T}_{\hbar k_0}\,|x\rangle\langle x|\psi\rangle$

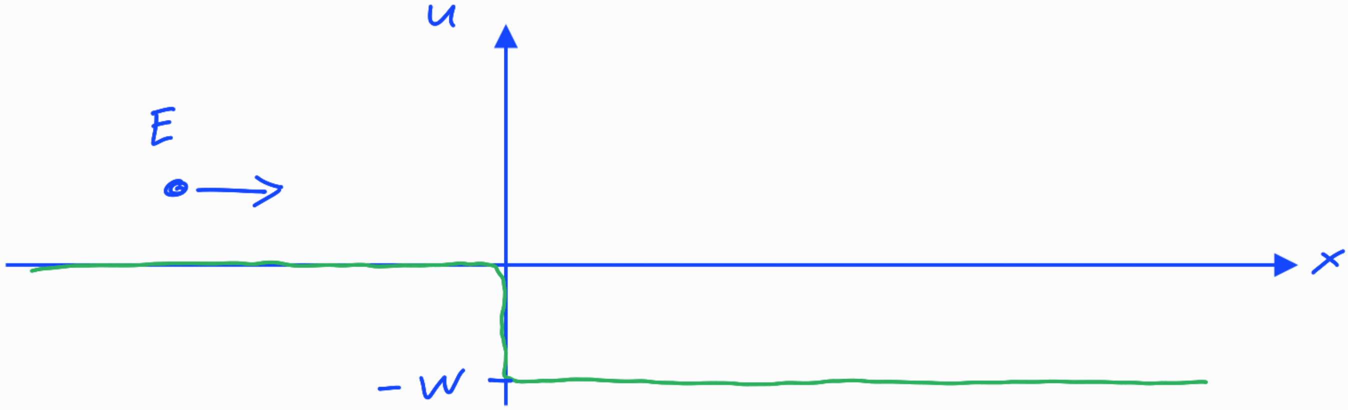
$$\overset{a)}{=} \int dx\; e^{ik_0 x}\,|x\rangle\langle x|\psi\rangle$$

$$= \int dx\; \underbrace{e^{ik_0 x}\,\psi(x)}_{\hat{=}\text{ Wellenfkt. von } \tilde{T}_{\hbar k_0}|\psi\rangle}\,|x\rangle \quad .$$

e) $T_{x_0}\,|\psi\rangle = \int \frac{dk}{2\pi}\; e^{-ix_0\hat{p}/\hbar}\,|\tilde{\varphi}_k\rangle\langle\tilde{\varphi}_k|\psi\rangle$

$$= \int dk\; \underbrace{e^{-ix_0 k}\;\tilde{\psi}(k)}_{\hat{=}\text{ Impulswellenfkt. von } T_{x_0}|\psi\rangle}\,|\tilde{\varphi}_k\rangle$$

25)



Streuansatz:

$$\psi(x) = \begin{cases} e^{ikx} + r\, e^{-ikx} & : x < 0 \\ t\, e^{ik'x} & : x \geq 0 \end{cases}$$

mit $k = \sqrt{2mE}/\hbar$, $k' = \sqrt{2m(E+W)}/\hbar$

Stetigkeit von ψ in $x=0$: $1 + r = t$

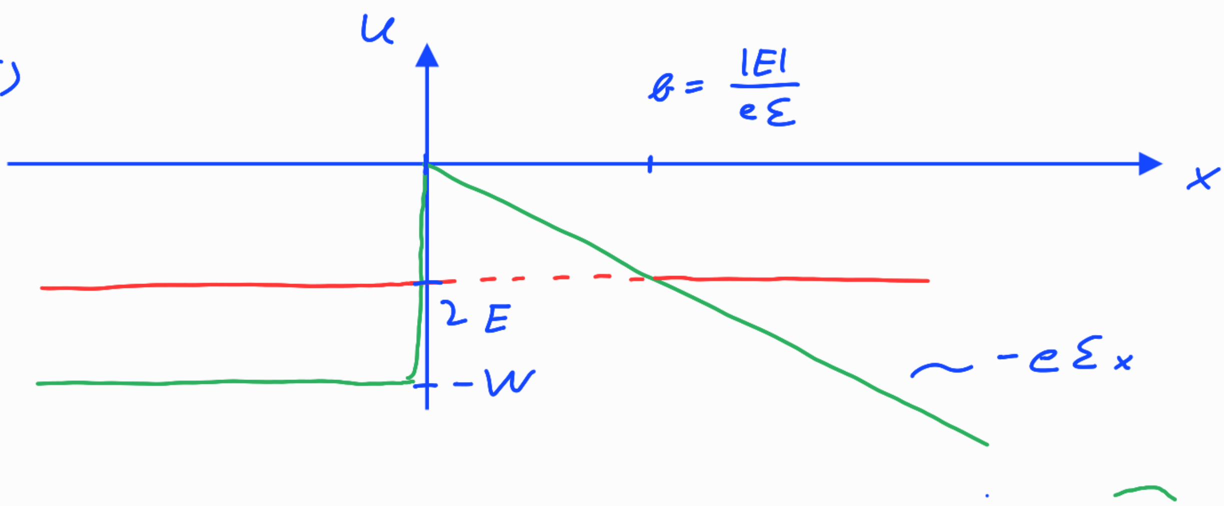
" " ψ' " " : $k(1-r) = k't$

$$\left.\begin{matrix} 1+r = t \\ k(1-r) = k't \end{matrix}\right\} \rightarrow 1 - r = \frac{k'}{k}(1+r)$$

$$\rightarrow \quad r = \frac{1 - k'/k}{1 + k'/k} = \frac{1 - \sqrt{1 + W/E}}{1 + \sqrt{1 + W/E}} \underset{W/E = 80}{=} -8/10$$

$$\rightarrow \quad R = |r|^2 = 0.64 \; .$$

26)



$$\frac{1}{\hbar}\int_0^b \left(8m(U(x)-E)\right)^{1/2} dx = \frac{\sqrt{8m}}{\hbar}\int_0^b \left(|E|-e\mathcal{E}x\right)^{1/2} dx$$

$$= -\frac{\sqrt{8m}}{\hbar}\frac{2}{3e\mathcal{E}}\left(|E|-e\mathcal{E}x\right)^{3/2}\Big|_0^b = \frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{|E|^{3/2}}{e\mathcal{E}}$$

$$\rightarrow \quad I(E,\mathcal{E}) \simeq I_0 \exp\left(-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{|E|^{3/2}}{e\mathcal{E}}\right)$$

signifikanten Tunnelstrom bei Feldstärke

$$\mathcal{E}_c \overset{!}{\simeq} \frac{4}{3}\frac{\sqrt{2m_e}}{\hbar}\frac{|E|^{3/2}}{e} \simeq 10^{10}\ V/m \ .$$

27)

$$|\psi(t)\rangle = \frac{e^{-i\omega t/2}}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

$$\rightarrow \langle a\rangle_t = \frac{1}{2}\left(\left(\langle 0| + e^{i\omega t}\langle 1|\right) a \left(|0\rangle + e^{-i\omega t}|1\rangle\right)\right)$$

$$= \frac{e^{-i\omega t}}{2}\underbrace{\langle 0|a|1\rangle}_{=1} = \frac{e^{-i\omega t}}{2}$$

$$\hookrightarrow \langle a^+\rangle_t = \langle a\rangle_t^* = \frac{e^{+i\omega t}}{2}$$

$$\hookrightarrow \langle x\rangle_t = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\langle a^+ + a\rangle_t = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\cos\omega t$$

$$\langle p\rangle_t = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i\langle a^+ - a\rangle_t = -\left(\frac{\hbar m\omega}{2}\right)^{1/2}\sin\omega t \ .$$