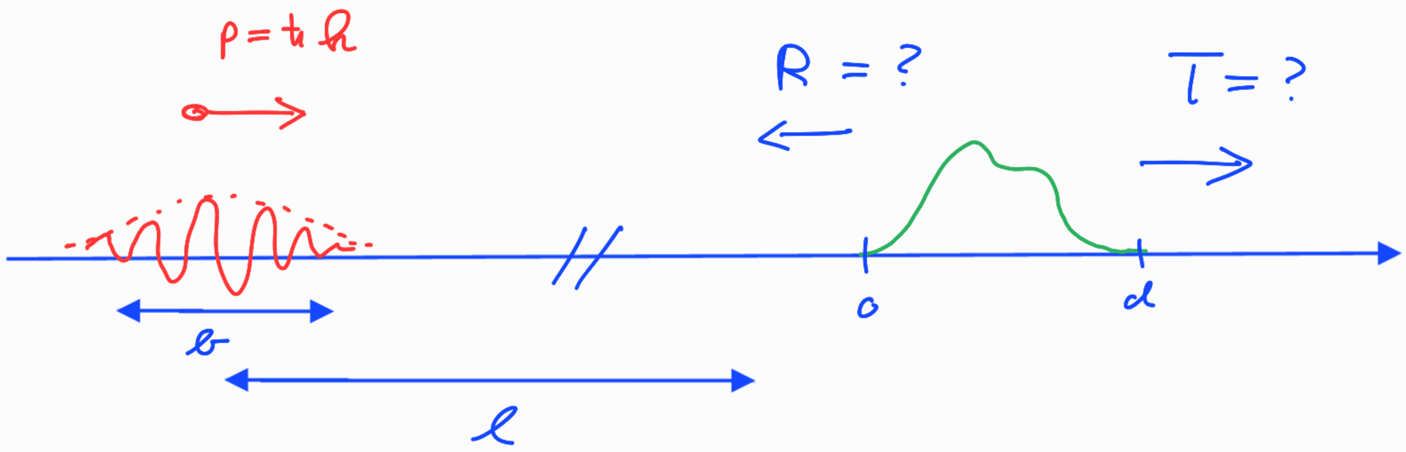
letzte Woche: Streuung an Potenzialschwelle



$\ell \gg b \gg d$

stationärer Streuansatz:

$$\psi(x) = \begin{cases} e^{ikx} + r\, e^{-ikx} & : \quad x \leq 0 \\ \psi_0(x) & : \quad 0 < x < d \\ t\, e^{ikx} & : \quad x \geq d \end{cases}$$

$$\psi_0''(x) = \frac{2m}{\hbar^2}\left(U(x) - E\right)\psi_0(x)$$

$\hookrightarrow$ s, u

r, t, s, u bestimmt durch Stetigkeit von ψ, ψ' bei $x = 0$, $x = d$

$\rightarrow$ Reflexionskoeff. $R = |r|^2$

Transmissionskoeff. $T = |t|^2$ $\qquad (U(\pm\infty) = 0)$

Whts.stromdichte: $j(x,t)$

$$\boxed{\underbrace{\frac{d}{dt}|\psi(x,t)|^2}_{\text{S.Gl.!}} + \underset{\underset{\frac{\partial}{\partial x}}{\shortparallel}}{\mathrm{div}}\, j(x,t) \overset{!}{=} 0} \qquad (1D)$$

$$\rightarrow \quad \boxed{j(x,t) = \frac{\hbar}{m}\,\mathrm{Im}\,\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)}$$

$$\left(3D:\ \vec{j} = \frac{\hbar}{m}\,\mathrm{Im}\,\psi^*\,\mathrm{grad}\,\psi\right)$$

Bemerkung:

Impulszustand $|\tilde{\varphi}_k\rangle$: Wfkt.: $\psi = e^{ikx}$

Dichte: $\rho = |\psi|^2 = 1$

Geschwindigkeit $v = \frac{\hbar k}{m}$ ← Impuls: $\hbar k$

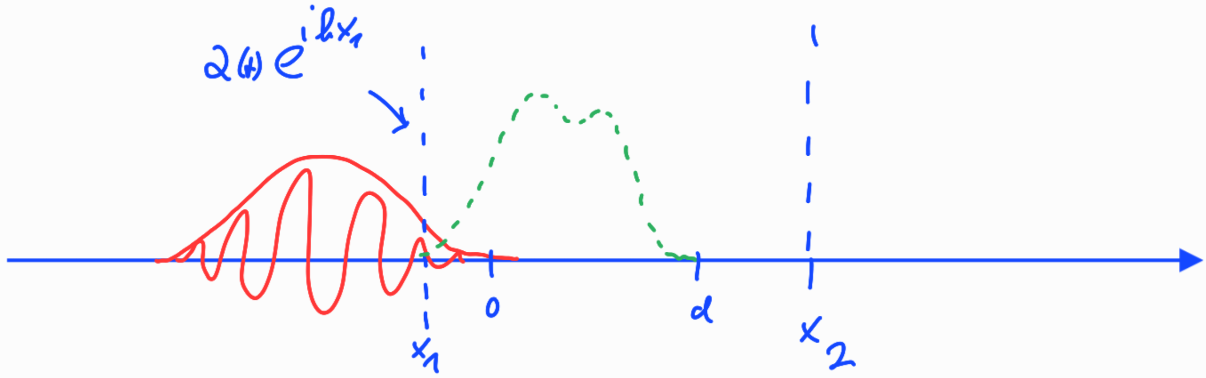
↳ Stromdichte $j = v\rho = \frac{\hbar k}{m} = \frac{\hbar}{m}\,\mathrm{Im}\,\underset{\underset{\psi^*}{\shortparallel}}{\underbrace{e^{-ikx}}}\frac{\partial}{\partial x}\underset{\underset{\psi}{\shortparallel}}{\underbrace{e^{ikx}}}$

$$\psi_1(x) = e^{ikx} + \underline{r}\, e^{-ikx} \rightarrow j_1(x) = \frac{p}{m}\left(1 - |r|^2\right)$$

$$\psi_2(x) = \underline{t}\, e^{ikx} \rightarrow j_2(x) = \frac{p}{m}\, |t|^2$$

$p = \hbar k$

$$r,\, t \quad \overset{?}{\longleftrightarrow} \quad R,\, T$$



ohne Barriere:

$$j_u(x_1 t) \overset{!}{=} f(t)\, \frac{\hbar k}{m} \;; \qquad \int_{-\infty}^{\infty} dt\; j_u(x_1, t) \overset{!}{=} 1$$

$\rightarrow$ Normierung $$\int_{-\infty}^{\infty} f(t)\, \frac{\hbar k}{m}\, dt = 1$$

mit Barriere:

$$j_u(x_1, t) = f(t)\, \frac{\hbar k}{m}\, \left(1 - |r|^2\right)$$

und

$$1 - R \overset{!}{=} \int_{-\infty}^{\infty} dt\; j_u(x_1, t) = \underbrace{\int_{-\infty}^{\infty} dt\, f(t)\, \frac{\hbar k}{m}}_{=1} \left(1 - |r|^2\right)$$

d. h. $R = |r|^2$!

analog:

$$T \overset{!}{=} \int_{-\infty}^{\infty} dt \; j_{\underline{\underline{u}}}(x_2, t) = \underbrace{\int_{-\infty}^{\infty} dt \; f(t) \frac{\hbar k}{m}}_{\hat{=}\, 1} |t|^2$$

d. h. $T = |t|^2$

Vorsicht: $U(-\infty) \neq U(+\infty)$

$$\to \quad \psi(x>a) = t \, e^{i \underline{\underline{k'}} x}$$

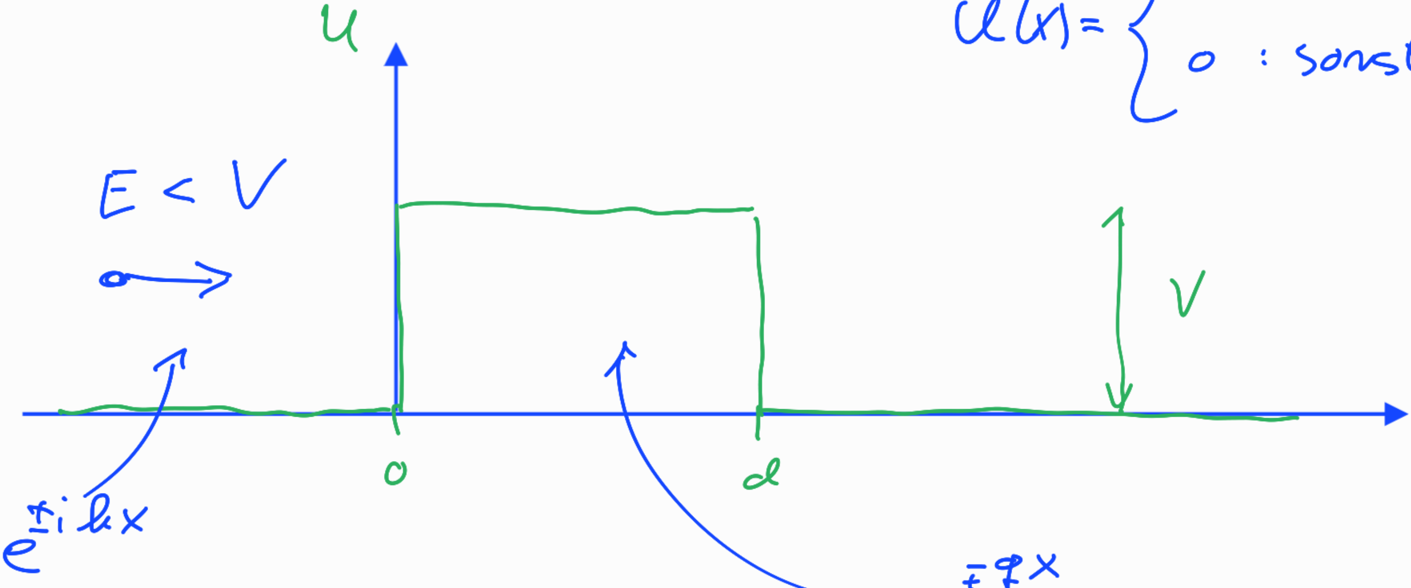
$$\neq$$

$$k$$

$\to$ für $x > a$: $j = \underline{\underline{\frac{\hbar k'}{m}}} |t|^2 \neq \frac{\hbar k}{m} |t|^2$!

Beispiel: "rechteckige" Potentialschwelle:

$$U(x) = \begin{cases} V & : 0 < x < d \\ 0 & : \text{sonst} \end{cases}$$



$$k = \sqrt{2mE}/\hbar \qquad q = \sqrt{2m(V-E)}/\hbar$$

Streuansatz:

$$\psi(x) = \begin{cases} e^{ikx} + \underline{\underline{r}}\, e^{-ikx} & : x \le 0 \\ \underline{\underline{s}}\, e^{-qx} + \underline{\underline{u}}\, e^{+qx} & : 0 < x < d \\ \underline{\underline{t}}\, e^{ik(x-d)} & : x \ge d \end{cases}$$

ψ, ψ' stetig in $x=0$, $x=d$!

$\rightarrow$ r, t, s, u !

$$x=0: \quad \left.\begin{aligned} 1 + r &= s + u \\ ik - ikr &= -qs + qu \end{aligned}\right\} \quad (i)$$

$$x=d: \quad \left.\begin{aligned} e^{-qd}s + e^{+qd}u &= t \\ -q e^{-qd}s + q e^{+qd}u &= ik \cdot t \end{aligned}\right\} \quad (ii)$$

$$\Longleftrightarrow$$

$$(i) \quad \overbrace{\begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}}^{A} \begin{pmatrix} 1 \\ r \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 \\ -q & q \end{pmatrix}}^{B} \begin{pmatrix} s \\ u \end{pmatrix}$$

$$(ii) \quad \underbrace{\begin{pmatrix} e^{-qd} & e^{+qd} \\ -q e^{-qd} & q e^{+qd} \end{pmatrix}}_{G} \begin{pmatrix} s \\ u \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ ik \end{pmatrix}}_{v} t$$

$$\Longleftrightarrow \quad \boxed{\begin{pmatrix} 1/t \\ r/t \end{pmatrix} \overset{!}{=} A^{-1} B G^{-1} v} \qquad !$$

per Computer-Algebra (Mathematica)

$$\boxed{\frac{1}{t} = \cosh(qd) + \frac{i}{2}\left(\frac{q}{k} - \frac{k}{q}\right) \sinh(qd)}$$

$d \to 0: \quad t = 1$ ✓

$d \to \infty: \quad t = 0$ ✓

$$\boxed{\frac{1}{t} = \cosh(qd) + \frac{i}{2}\left(\frac{q}{k} - \frac{k}{q}\right) \sinh(qd)}$$

$$\hookrightarrow \quad T^{-1} = \frac{1}{|t|^2} = \underbrace{\cosh^2(qd)}_{1+\sinh^2(qd)} + \frac{1}{4}\left(\frac{q}{k} - \frac{k}{q}\right)^2 \sinh^2(qd)$$

$$\rightarrow \quad \boxed{T = \left(1 + \left\{1 + \frac{1}{4}\left(\frac{q}{k} - \frac{k}{q}\right)^2\right\} \sinh^2(qd)\right)^{-1}}$$

Grenzfall: $\quad qd \gg 1 \quad \Leftrightarrow$

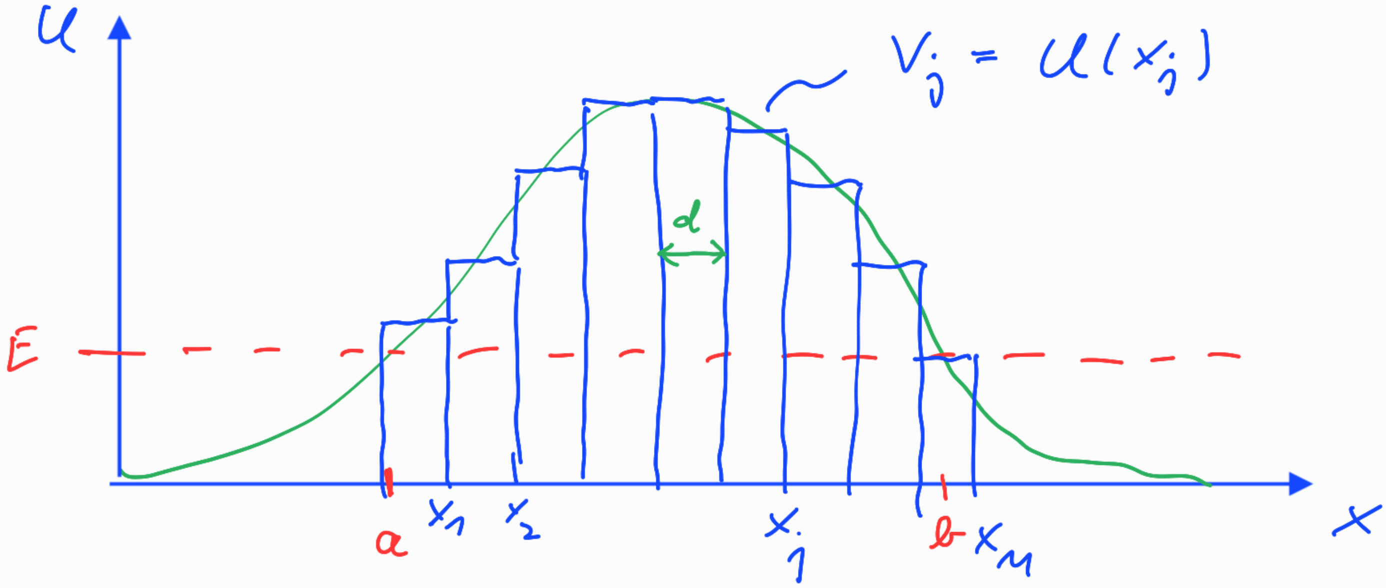
$$d \gg \frac{1}{q} = \frac{\hbar}{\sqrt{2m(V-E)}}$$

$$\rightarrow \quad \sinh(qd) = \frac{1}{2}\left(e^{qd} - \cancel{e^{-qd}}\right) = e^{qd}/2$$

$$\rightarrow \quad T = \frac{1}{\underbrace{4\{\dots\}}} \, e^{-2qd}$$

$$\boxed{T \simeq e^{-2qd} = e^{-d/\lambda} \qquad \text{mit} \quad \lambda = \frac{1}{2q} = \hbar / \sqrt{8m(V-E)}}$$

Abschätzung der Tunnelwkt. für bel. $U(x)$:



$$V_j \rightarrow q_j = \sqrt{2m(U(x_j)-E)}/\hbar$$

$$d_j = d$$

$$\rightarrow \quad T_j = e^{-2q_j d}$$

T = Wkt, dass alle Barriere durchtunnelt werden:

$$T = T_1 \cdot T_2 \cdot \ldots T_M = \exp\left(-\underbrace{\sum_{j=1}^{d} 2q_j d}\right)$$

$$\rightarrow \sum_{j=1}^{d} 2q_j d = \frac{1}{\hbar}\sum_{j=1}^{d} \underbrace{\sqrt{8m(U(x_j)-E)}} \cdot d$$

$$= \frac{1}{\hbar}\int_a^b \left(8m(U(x)-E)\right)^{1/2} \cdot dx \quad !$$

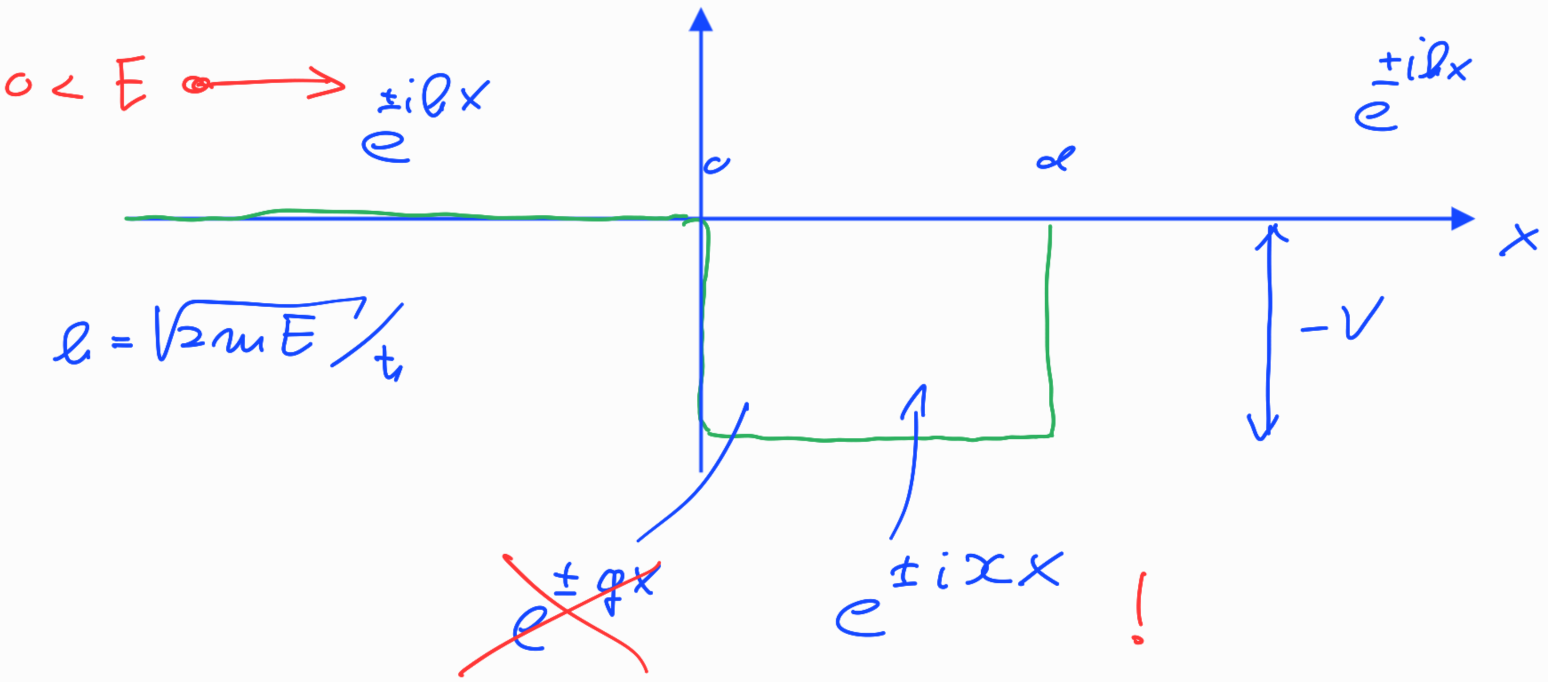
„$d \rightarrow 0$", (U hinreichend glatt)

$$\rightarrow \quad T \simeq \exp\left(-\frac{1}{\hbar}\int_a^b \sqrt{8m(U(x)-E)}\,dx\right)$$

Transmissionsfaktor im Gamow-Näherung

(genauer mittels: WKB-Methode)

2. Beispiel: Streuung am Potentialtopf



$$\varkappa = \sqrt{2m(E+V)}/\hbar \quad !$$

dieselbe Rechnung: $\quad q \rightarrow i\varkappa$!

altes Resultat: $\quad (\quad V > E > 0)$:

$$1/t_{alt} = \cosh(qd) + \frac{i}{2}\left(\frac{q}{k} - \frac{k}{q}\right)\sinh(qd)$$

$q \to i\varkappa$

$$1/t = \underbrace{\cosh(i\varkappa d)}_{\cos(\varkappa d)} + \frac{i}{2}\left(\frac{i\varkappa}{k} + \frac{ik}{\varkappa}\right)\underbrace{\sinh(i\varkappa d)}_{i\sin(\varkappa d)}$$

$$T^{-1} = |t|^{-2} = \underbrace{\cos^2(\varkappa d)}_{1-\sin^2(\varkappa d)} + \frac{1}{4}\left(\frac{\varkappa}{k} + \frac{k}{\varkappa}\right)^2 \sin^2(\varkappa d)$$

$$= 1 + \underbrace{\left\{-1 + \frac{1}{4}\left(\frac{\varkappa}{k} + \frac{k}{\varkappa}\right)^2\right\}}_{\frac{1}{4k^2\varkappa^2}\left\{\underbrace{(\varkappa^2+k^2)^2 - 4k^2\varkappa^2}_{(\varkappa^2-k^2)^2}\right\}} \sin^2(\varkappa d)$$

$$\to \boxed{T = \left(1 + \left(\frac{\varkappa^2 - k^2}{2k\varkappa}\right)^2 \sin^2(\varkappa d)\right)^{-1}}$$

für $\varkappa d \overset{!}{=} \pi n$: $\quad T = 1$!

T minimal für $\varkappa d = \frac{\pi}{2}(2n+1)$!

$$\rightarrow \boxed{T = \left(1 + \left(\frac{\varkappa^2 - k^2}{2k\varkappa} \right)^2 \sin^2(\varkappa d) \right)^{-1}}$$

für $\varkappa d \overset{!}{=} \pi n$: $T = 1$!

T minimal für $\varkappa d = \frac{\pi}{2}(2n+1)$!