Letzte Vrlsg.: ("Drehimpuls $\equiv$ Erzeuger der Rotation")

$R : \mathbb{R}^3 \to \mathbb{R}^3$ Rotation g.d.w.: • $R^T R = \mathbb{1}_3$

$\vec{r} \mapsto R\vec{r}$ • $\det R = 1$

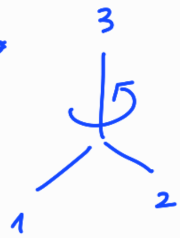
$\rightarrow$ speziell-orthogonale Gruppe (Drehgruppe)

$$SO(3) = \{ R \in M(3\times 3, \mathbb{R}) \mid R^T R = 1, \det R = 1 \}$$

zugleich 3-dim. Mannigfaltigkeit (d.h. 3 konti. Param.: z.B. $\vec{n}, \varphi \to R_{\vec{n},\varphi}$ $|\vec{n}|=1$)

damit $SO(3)$ auch Lie-Gruppe

$\rightarrow$ $SO(3)$ weitgehend bestimmt durch infinitesimale Rotationen $R \simeq \mathbb{1}_3$:

• 

$$R_{3,\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{\varphi \ll 1}{=} \mathbb{1}_3 + \varphi I_3$$

mit $I_3 = \left.\dfrac{\partial R_{3,\varphi}}{\partial \varphi}\right|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

analog: $$I_1 = \frac{\partial R_{1,\varphi}}{\partial \varphi}\bigg|_{\varphi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I_2 = \frac{\partial R_{2,\varphi}}{\partial \varphi}\bigg|_{\varphi=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$\to$ • $$R_{\ell,\varphi} = e^{I_\ell \varphi}$$

• $$R_{\vec{u},\varphi} = e^{\vec{u} \cdot \vec{I} \varphi}$$

allg. Rotat.

$$\vec{u} \cdot \vec{I} = u_1 I_1 + u_2 I_2 + u_3 I_3 = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$$

$\to$ $\mathrm{Span}\{I_1, I_2, I_3\}$ = VR V der inf. Erzeuger!

$$= \{ I \in M(3\times 3, \mathbb{R}) \mid I^T = -I \}$$

abgeschloss. bzgl. $[\ldots,\ldots]$!

$(V, [\ldots,\ldots]) \equiv$ <u>Lie-Algebra</u> $so(3)$

der Lie-Gruppe $SO(3)$

<u>Vertauschungsrelationen</u> in $so(3)$ bestimmen $SO(3)$:

$$[I_1, I_2] = I_3, \quad [I_2, I_3] = I_1, \quad [I_3, I_2] = I_1$$

d. h.

$$\boxed{[\, I_k, I_l \,] = \varepsilon_{klm} I_m}$$

z. B. $[I_1, I_2] = \underbrace{\begin{pmatrix} 0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0\end{pmatrix}}_{= \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0\end{pmatrix}} - \underbrace{\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\begin{pmatrix} 0&0&0\\0&0&-1\\0&1&0\end{pmatrix}}_{= \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0\end{pmatrix}}$

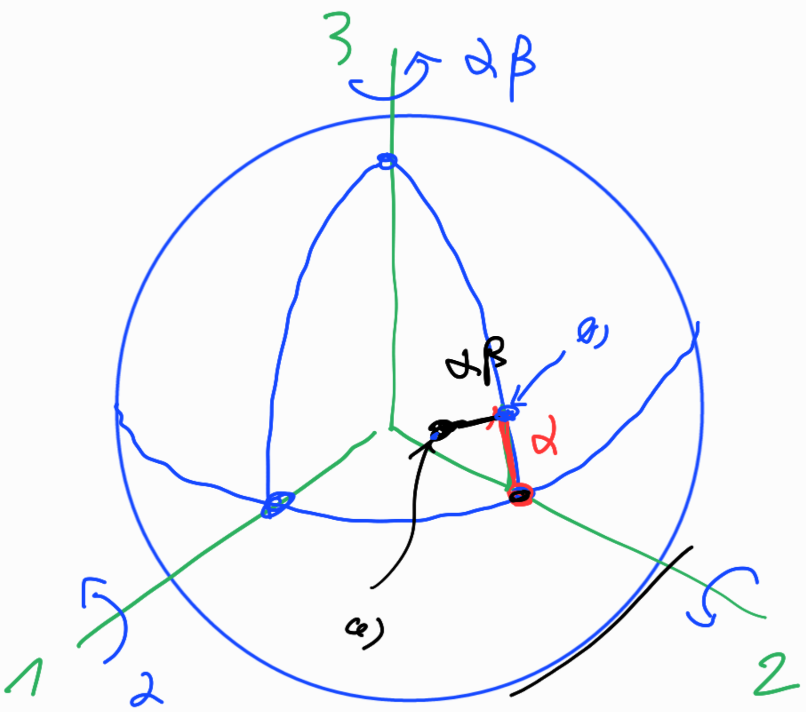
$$= \begin{pmatrix} 0&-1&0\\1&0&0\\0&0&0\end{pmatrix} = I_3 \,.$$

$\perp$

geometrische Bedeutung von z. B.

$$[\, I_1, I_2 \,] = I_3 \quad ?$$

$\leadsto \quad I_1 I_2 = I_2 I_1 + I_3$



a) $\quad R_{2,\beta} \, R_{1,\alpha}$

b) $\quad R_{1,\alpha} \, R_{2,\beta}$

$$\boxed{R_{1,\alpha} R_{2,\beta} \overset{!}{=} R_{3,\alpha\beta} R_{2,\beta} R_{1,\alpha} + \mathcal{O}(\alpha,\beta)^3} \quad (*)$$

ND: Baker-Campbell-Hausdorff-Formel:

A, B, vertauschen mit $[A,B]$

$$\rightarrow \quad e^A e^B \overset{!}{=} e^{\frac{1}{2}[A,B]} e^{A+B} = e^{[A,B]} e^B e^A$$

$$\overset{\shortparallel}{e^{B+A}} = e^{-\frac{1}{2}\underbrace{[B,A]}_{-[A,B]}} e^B e^A$$

$$\boxed{e^A e^B = e^{[A,B]} e^B e^A}$$

$$\underbrace{R_{1,\alpha}}_{e^{I_1\alpha}} \underbrace{R_{2\beta}}_{e^{I_2\beta}} = e^{\alpha\beta \overbrace{[I_1,I_2]}^{=I_3}} \underbrace{e^{I_2\beta}}_{R_{2,\beta}} \underbrace{e^{I_1\alpha}}_{R_{1,\alpha}}$$

$$e^{I_1\alpha} e^{I_2\beta} = R_{3,\alpha\beta} R_{2,\beta} R_{1,\alpha} + \mathcal{O}(\alpha,\beta)^3$$

Übersetzung in die QM:

$SO(3) \ni R \longmapsto \mathcal{U}(R)$: operiert auf $\mathcal{H}$

$\vec{r} \mapsto R\vec{r}$

$|\psi\rangle \mapsto \mathcal{U}(R)|\psi\rangle$!

Anforderungen:

(1)
$$\boxed{\begin{aligned} &\bullet \quad \mathcal{U}(R) \ \text{unitär} \\ &\bullet \quad \mathcal{U}(\mathbb{1}_3) = \mathbb{1}_{\mathcal{H}} \\ &\bullet \quad \mathcal{U}(R'R) \overset{!}{=} \mathcal{U}(R')\,\mathcal{U}(R) \end{aligned}}$$!

⌈ Mathe: $R \longmapsto \mathcal{U}(R)$ ist Darstellung der Gruppe $SO(3)$ durch Op. auf $\mathcal{H}$ ⌋

$$\boxed{\mathcal{U}_{\text{ort}}(R)\,|\vec{r}\rangle \overset{!}{=} |R\vec{r}\rangle} \qquad (2)$$

$(T(\vec{a})|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle$

$\rightarrow$ $\mathcal{U}(R)$ Rotation auf $\mathcal{H}$!

$\rightarrow$ Drehimpulse $\vec{J} = (J_1, J_2, J_3)$

$$\boxed{J_\ell := i\hbar \frac{\partial}{\partial\varphi}\, \mathcal{U}(R_{\ell,\varphi})\Big|_{\varphi=0}}$$

(vgl.: $P_\ell := i\hbar \frac{\partial}{\partial a} T(a\vec{e}_\ell)$)

wir zeigen: $\boxed{[J_l, J_k] = i\hbar\, \varepsilon_{lkm} J_m}$ (a)

$\vec{J} = \vec{L} + \vec{S}$: $[L_l, L_k] = i\hbar\, \varepsilon_{lkm} L_m$

$[S_l, S_k] = i\hbar\, \varepsilon_{lkm} S_m$

(b)

$$\boxed{U(R_{\vec{n},\varphi}) = e^{-i\vec{n}\cdot\vec{J}\varphi/\hbar} = e^{-i\vec{n}\cdot\vec{L}\varphi/\hbar} \otimes e^{-i\vec{n}\cdot\vec{S}\varphi/\hbar}}$$

$(\vec{n}\cdot\vec{J} = n_1 J_1 + n_2 J_2 + n_3 J_3)$

mit (2) folgt für Bahndrehimpuls:

$$\boxed{\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}} \quad (c)$$

zu a)

benötigen: $[I_1, I_2] \overset{!}{=} I_3$ (+ zykl.)

$\hookrightarrow \quad R_{1,\alpha} R_{2,\beta} = R_{3,\alpha\beta} R_{2,\beta} R_{1,\alpha} + O(\alpha,\beta)^3$

zu a)

benötigen: $\underline{[I_1, I_2] \overset{!}{=} I_3}$ (+ zykl.)

$\Leftrightarrow \quad R_{1,\alpha} R_{2,\beta} \overset{!}{=} R_{3,\alpha\beta} R_{2,\beta} R_{1,\alpha} + \underline{O(\alpha,\beta)^3}$

$$\underline{J_1 J_2} = \left(i\hbar\frac{\partial}{\partial\beta}\right)\left(i\hbar\frac{\partial}{\partial\alpha}\right) \underbrace{U(R_{1,\alpha})\, U(R_{2,\beta})}_{U(\underbrace{R_{1,\alpha} R_{2,\beta}}_{\overset{(*)}{=}})}\Big|_{\underline{\alpha,\beta=0}}$$

$$U(\underline{R_{3,\alpha\beta}}\, \underline{R_{2,\beta} R_{1,\alpha}})$$

$$= \left(i\hbar\frac{\partial}{\partial\alpha}\right)\underbrace{\left(i\hbar\frac{\partial}{\partial\beta}\, U(R_{3,\alpha\beta})\, U(R_{2,\beta})\right)}_{(\alpha J_3 + J_2)} \underline{U(R_{1,\alpha})}$$

$$= \underline{i\hbar J_3} + \underline{J_2 J_1}$$

$$[J_1, J_2] = i\hbar J_3 \,. \qquad \text{(Rest analog!)}$$

zu (b):

$$\boxed{U(R_{\vec{n},\varphi}) \overset{!}{=} e^{-i\vec{n}\cdot\vec{J}\varphi/\hbar}}$$

$$\left(T(\vec{a}) = e^{-i\vec{a}\cdot\hat{\vec{p}}/\hbar} \right)$$

wir zeigen:

$$\frac{\partial}{\partial\varphi} \mathcal{U}(R_{\vec{n},\varphi})\Big|_{\varphi_0} \overset{!}{=} -\frac{i}{\hbar}\, \vec{n}\cdot\vec{J}\; \mathcal{U}(R_{\vec{n},\varphi_0})$$

$$\left(\dot{y} = a\, y \quad \to \quad y(t) = e^{at} \right)$$

$$\hookrightarrow \quad (8) \;!$$

l.S.

$$\frac{\partial}{\partial\varphi} \underbrace{\mathcal{U}(R_{\vec{n},\varphi} R_{\vec{n},\varphi_0})}_{\mathcal{U}(R_{\vec{n},\varphi})\, \mathcal{U}(R_{\vec{n},\varphi_0})}\Big|_{\varphi=0}$$

$$= \frac{\partial}{\partial\varphi} \mathcal{U}\big(e^{\vec{I}\cdot\vec{n}\varphi} \big)\, \mathcal{U}(R_{n,\varphi_0})$$

$$e^{\vec{I}\cdot\vec{n}\varphi} = e^{I_1 n_1 \varphi}\, e^{I_2 n_2 \varphi}\, e^{I_3 n_3 \varphi}$$

$$= \frac{\partial}{\partial\varphi} \mathcal{U}(R_{1,n_1\varphi})\, \mathcal{U}(R_{2,n_2\varphi})\, \mathcal{U}(R_{3,n_3\varphi})\Big|_{\varphi=0} \cdot \mathcal{U}(R_{\vec{n},\varphi_0})$$

$$= -\frac{i}{\hbar} \underbrace{(n_1 J_1 + n_2 J_2 + n_3 J_3)}_{\vec{n}\cdot\vec{J}\;!}\, \mathcal{U}(R_{\vec{n},\varphi_0})$$

zu (c): $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

z.B. $\hat{L}_3 \overset{!}{=} \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1 \overset{!}{=} i\hbar \frac{\partial}{\partial\varphi} U_{\text{ort}}(R_{3,\varphi})$

$$\hat{L}_3 |\vec{r}\rangle = i\hbar \frac{\partial}{\partial\varphi} \underbrace{U_{\text{ort}}(R_{3,\varphi}) |\vec{r}\rangle}_{\substack{\| \\ \underbrace{|R_{3,\varphi}\vec{r}\rangle}_{\underbrace{|(\mathbb{1}+\varphi I_3)\vec{r}\rangle}_{\left|\begin{pmatrix} x_1 - \varphi x_2 \\ x_2 + \varphi x_1 \\ x_3 \end{pmatrix}\right\rangle}}}} \Bigg|_{\varphi=0}$$

$$\hat{L}_3 |\vec{r}\rangle = i\hbar \frac{\partial}{\partial\varphi} T(-\varphi x_2 \vec{e}_1)\, T(\varphi x_1 \vec{e}_2) \Big|\vec{r}\rangle \Big|_{\varphi=0}$$

$$= \left(-x_2 \hat{p}_1 + x_1 \hat{p}_2\right) |\vec{r}\rangle$$

$$a \hat{p}_e = i\hbar \frac{\partial}{\partial\varphi} T(a\varphi \vec{e}_e)$$

$$\hat{L}_3 |\vec{r}\rangle \overset{!}{=} (-\hat{x}_2 \hat{p}_1 + \hat{x}_1 \hat{p}_2) |\vec{r}\rangle$$

$$\rightarrow \quad \hat{L}_3 = (\hat{\vec{r}} \times \vec{p})_3 \quad .$$

Transformation von $\mathcal{H}_{spin} = \text{Span}\{ |\uparrow\rangle = \binom{1}{0}, |\downarrow\rangle = \binom{0}{1} \}$

$$U_{spin}(R_{\vec{n},\varphi}) \underset{\beta)}{\overset{!}{=}} e^{-i\vec{n}\cdot\vec{S}\varphi/\hbar}$$

wie lauten die Spinop. S_1, S_2, S_3 ?

genügen: $[S_l, S_k] \overset{!}{=} i\hbar\, \varepsilon_{lkm} S_m$

Pauli-Operator! $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

erfüllen: $[\sigma_l, \sigma_k] \overset{!}{=} 2i\, \varepsilon_{lkm}\sigma_m \quad \Big|\ \frac{\hbar^2}{4}$

$$\left[\frac{\hbar}{2}\sigma_l, \frac{\hbar}{2}\sigma_k\right] = i\hbar\, \varepsilon_{lkm} \frac{\hbar}{2}\sigma_m$$

$$\rightarrow \boxed{S_l = \frac{\hbar}{2}\sigma_l} \rightarrow \vec{S} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

$$\rightarrow U_{spin}(R_{\vec{n},\varphi}) = e^{-i\vec{n}\vec{S}\varphi/\hbar}$$

$$= e^{-i\vec{n}\cdot\vec{\sigma}\frac{\varphi}{2}}$$

$$\overset{!}{=} \mathbb{1}\cos\frac{\varphi}{2} - i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\varphi}{2}$$

$|\vec{n}| = 1$

$(\vec{n}\cdot\vec{\sigma})^2 = \mathbb{1}_2$

$\rightarrow$ EZ und EW von S_e :

S_1 : EZ $|x\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$; EW: $\pm\frac{\hbar}{2}$

S_2 : " $|y\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$, EW: $\pm\frac{\hbar}{2}$

S_3 : $|z+\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$,

$|z-\rangle = \begin{pmatrix}0\\ -1\end{pmatrix}$, EW: $\pm\frac{\hbar}{2}$