Letzte Vrlsg:

$$SO(3) \ni R \longmapsto \mathcal{U}(R) \in \mathcal{L}(\mathcal{H})$$

$$\vec{r} \mapsto R\vec{r} \qquad |\psi\rangle \mapsto \mathcal{U}(R)|\psi\rangle$$

s.d.:

- $\mathcal{U}(R)$ unitär
- $\mathcal{U}(\mathbb{1}_3) = \mathbb{1}_{\mathcal{H}}$
- $\mathcal{U}(R'R) = \mathcal{U}(R')\,\mathcal{U}(R)$

⌈ $R \mapsto \mathcal{U}(R)$ Darstellung der $SO(3)$ ⌋

- $\mathcal{U}_{Ort}(R)\,|\vec{r}\rangle = |R\vec{r}\rangle$

→ Drehimpuls

$$\boxed{J_e := i\hbar \frac{\partial}{\partial\varphi}\,\mathcal{U}(R_{e,\varphi})\Big|_{\varphi=0}}$$

→ Vertauschungsrelationen:

$$\boxed{[J_k, J_l] = i\hbar\,\varepsilon_{klm} J_m}$$

⌈ ebenso für L_k, S_k da $\vec{J} = \vec{L} + \vec{S}$ ⌋

$$\rightarrow\ \mathcal{U}(R_{\vec{n},\varphi}) = e^{-i\vec{n}\cdot\vec{J}\varphi/\hbar} = e^{-i\vec{n}\cdot\vec{L}\varphi/\hbar} \otimes e^{-i\vec{n}\cdot\vec{S}\varphi/\hbar}$$

→ Bahndrehimpuls: $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$

Spinoperatoren

$$\vec{S} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

(= Drehimpulsoperatoren des "Eigendrehimpulses")

- genügen $\boxed{[S_k, S_l] \overset{!}{=} i\hbar\, \varepsilon_{klm} S_m}$
- operieren auf $\mathcal{H}_{spin} = \mathrm{Span}\{ |\uparrow\rangle = \binom{1}{0}, |\downarrow\rangle = \binom{0}{1} \} \cong \mathbb{C}^2$

$\Rightarrow$ d.h. $\boxed{S_k \in M(2\times 2, \mathbb{C})}$

$\rightarrow$ $\boxed{S_k = \frac{\hbar}{2} \sigma_k}$ (eindeutig bis auf unitär äquivalente Darstellungen, vgl. Übg.)

d.h. $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ mit $\vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$:

	$S_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$S_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$S_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Eigenzust.:	$\lvert x\pm\rangle = \frac{1}{\sqrt{2}}\binom{1}{\pm 1}$	$\lvert y\pm\rangle = \frac{1}{\sqrt{2}}\binom{1}{\pm i}$	$\lvert\uparrow\rangle = \lvert z+\rangle = \binom{1}{0}$ $\lvert\downarrow\rangle = \lvert z-\rangle = \binom{0}{1}$
Eigenwerte:	$\pm \hbar/2$	$\pm \hbar/2$	$\pm \hbar/2$

→ Transformation eines Spinzustandes

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad (\in \mathbb{C}^2)$$

unter Rotation $R_{\vec{u},\varphi}$ $\quad (\in SO(3))$:

$$|\psi\rangle \longmapsto |\psi'\rangle = \mathcal{U}_{spin}(R_{\vec{u},\varphi})\,|\psi\rangle \quad \text{mit}$$

$$\boxed{\mathcal{U}_{spin}(R_{\vec{u},\varphi}) = e^{-i\vec{u}\cdot\vec{S}\,\varphi/\hbar} = e^{-i\,\vec{u}\cdot\vec{\sigma}\,\frac{\varphi}{2}}}$$

$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

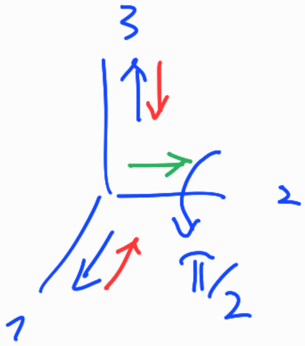
$$\vec{n}\cdot\vec{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 = \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix}$$

$$\rightarrow (\vec{u}\cdot\vec{\sigma})^2 = \mathbb{1}_2 \qquad (|\vec{n}| = 1)$$

$$\rightarrow \quad \boxed{\mathcal{U}_{spin}(R_{\vec{u},\varphi}) = \mathbb{1}\cos\varphi/2 - i\,\vec{u}\cdot\vec{\sigma}\,\sin\varphi/2}$$

Check:

$$U_{2,\pi/2} := U_{sp}\left(R_{2,\pi/2}\right) \quad , \quad -i\sigma_2 = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$



$$= \mathbb{1}\underbrace{\cos \pi/4}_{1/\sqrt{2}} + \begin{pmatrix}0&-1\\1&0\end{pmatrix}\underbrace{\sin \pi/4}_{1/\sqrt{2}}$$

$$U_{2,\pi/2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix} \; ;$$

- $U_{2,\pi/2}\,|z+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |x+\rangle$ ✓

  " " $|z-\rangle$ = " " $\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = -|x-\rangle$ ✓

- $U_{2,\pi/2}\,|x+\rangle = \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |z-\rangle$ ✓

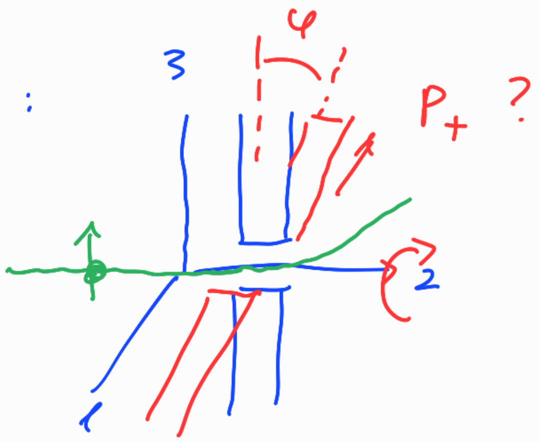
  " $|x-\rangle$ = " " $\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |z+\rangle$ ✓

- $U_{2,\pi/2}\,|y+\rangle = \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1-i\\1+i\end{pmatrix}$

$$= \frac{1-i}{2}\begin{pmatrix}1\\ \frac{1+i}{1-i}\end{pmatrix} = e^{-i\pi/4}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}}_{|y+\rangle} \quad ✓$$

" " $|y-\rangle$ = " " $= e^{+i\pi/4}\,|y-\rangle$ ✓

Beispiel:



$$|z+'\rangle = \mathcal{U}(R_{2,\varphi})\,|z+\rangle$$

$$= \mathbb{1}\cos\varphi/2 + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}}_{= -i\sigma_2} \sin\varphi/2 = \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix}$$

$$P_+ = \left|\langle z+' \,|\, z+\rangle\right|^2 = \left|\underset{\begin{pmatrix}1\\0\end{pmatrix}}{\langle z+|}\ \mathcal{U}_{2,\varphi}\ \underset{\begin{pmatrix}1\\0\end{pmatrix}}{|z+\rangle}\right|^2$$

$$= |\cos\varphi/2|^2$$

⌈ $\varphi = 30^\circ = \pi/6 \quad\rightarrow\quad P_+ = |\cos\pi/12|^2$

$$= 0{,}93\ .$$ ⌋

Bemerkung: $U(R_{\vec{u},\varphi}) = \mathbb{1}\cos\varphi/2 - i\,\vec{\sigma}\cdot\vec{u}\,\sin\varphi/2$

2π-Rot.: $U(R_{\vec{u},2\pi}) = -\mathbb{1}$ | $R_{\vec{u},2\pi} = \mathbb{1}$

$\neq$ (!!)

4π-Rot.: $U(R_{\vec{u},4\pi}) = +\mathbb{1}$ | $R_{\vec{u},4\pi} = \mathbb{1}$

(→ AQM, QFT)

Drehimpulseigenwerte und -zustände

Ausgangspkt.: $[J_k, J_l] = i\hbar\,\varepsilon_{klm} J_m$ (*)

→ allg. Aussagen über Eigenwerte / -zustände:

$$\underline{\underline{J^2 \text{ und } J_3}}$$

$\left[J^2 = J_1^2 + J_2^2 + J_3^2 \right.$

(*) → $[J^2, J_l] = \dots = 0$ $\lrcorner$

insb. $[J^2, J_3] = 0$

→ gemeinsame Eigenbasis von J^2, J_3:

Eigenbasis $|a, b\rangle$:

$$J^2 |a,b\rangle = a\, |a,b\rangle$$

$$J_3 |a,b\rangle = b\, |a,b\rangle$$

a, b ?

Hilfreich:

$$\boxed{\begin{aligned} J_+ &:= J_1 + i J_2 \\ J_- &:= J_1 - i J_2 \end{aligned}} \quad {}^+$$

(*) $\rightarrow$

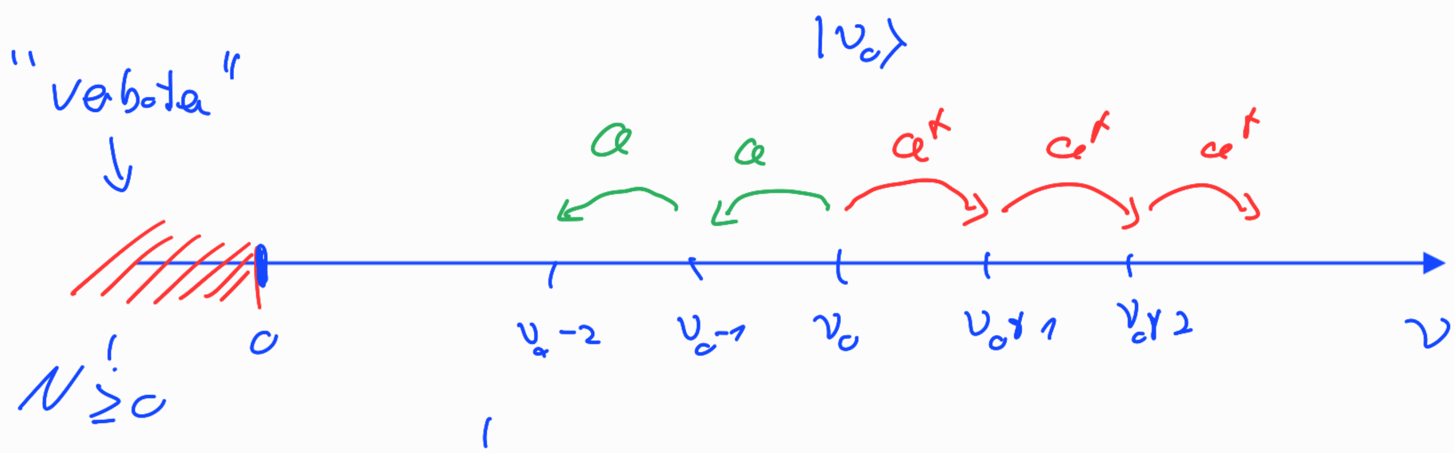
- $[J_3, J_+] = \hbar J_+$
- $[J_3, J_-] = -\hbar J_-$

(• $[J^2, J_\pm] = 0$)

[Erinnerung: harm. Oszill.: a, a^+ , $[a, a^+] = \mathbb{1}$

$N = a^+ a$, $\quad N|\nu\rangle = \nu |\nu\rangle$

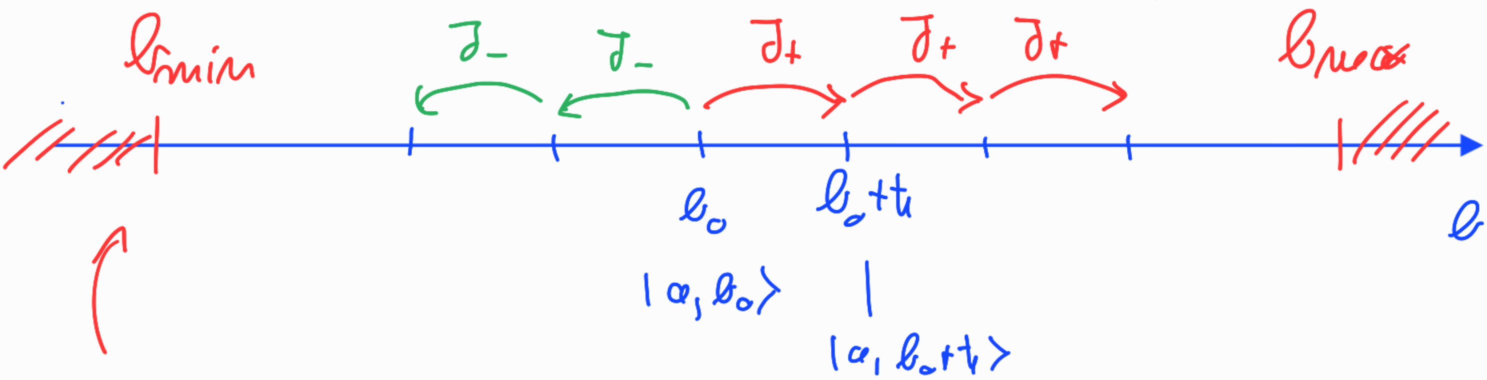
- $[N, a^+] = a^+$
- $[N, a] = -a$



$$a|0\rangle \overset{!}{=} 0 \quad \rightarrow \quad \nu_0 \in \mathbb{N}$$

$$\rightarrow \quad |\nu\rangle, \quad \nu = 0, 1, 2, 3, \ldots$$

mit gleicher Rechnung (aufgrund gleicher Komm.-Relationen):



(*)

$$\boxed{0 \overset{!}{\leq} J^2 - J_3^2}$$

$$\left[\ \langle J^2 - J_3^2\rangle_\psi = \langle J_1^2\rangle_\psi + \langle J_2^2\rangle_\psi \geq 0! \right.$$

für bel. $|\psi\rangle$ $\left.\right]$

$\rightarrow$ es ex. Zust. $|a, b_{max}\rangle$, $|a, b_{min}\rangle$ s. d.

- $$J_+ |a, b_{max}\rangle \overset{!}{=} 0$$
- $$J_- |a, b_{min}\rangle \overset{!}{=} 0$$

$$\underbrace{J_- J_+}_{\|} |a, b_{max}\rangle \overset{!}{=} 0$$

$$(J_1 - iJ_2)(J_1 + iJ_2) = \overbrace{J_1^2 + J_2^2}^{J^2 - J_3^2} + i\overbrace{[J_1, J_2]}^{iJ_3 \cdot \hbar}$$

$$= J^2 - J_3^2 - \hbar J_3$$

$$(J^2 - J_3^2 - \hbar J_3)|a, b_{max}\rangle = 0$$

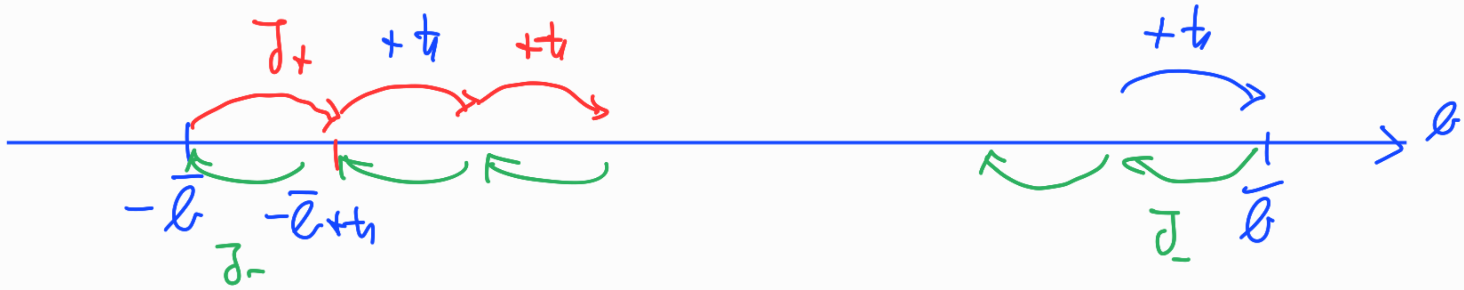
$$(a - b_{max}^2 - \hbar b_{max})|a, b_{max}\rangle = 0$$

d. h. $\cdot$ $a = b_{max}(b_{max} + \hbar)$

analog: $\cdot$ $a = b_{min}(b_{min} - \hbar)$

$\rightarrow$ $b_{min} = -b_{max} \equiv -\bar{b}$

$b_{max} =: \bar{b}$



$$\rightarrow \quad 2\bar{b} = \underline{n}\,\hbar \quad , \quad n = 0, 1, \dots$$

$$\bar{b} = \hbar \cdot \underline{\frac{n}{2}} = \left(0, \frac{\hbar}{2}, \hbar, \frac{3}{2}\hbar, \dots\right)$$

$$\rightarrow \quad a = \bar{b}\,(\bar{b} + \hbar)$$

$$= \hbar^2\, \underline{\frac{n}{2}}\left(\underline{\frac{n}{2}} + 1\right) \rightarrow j$$

Drehimpulsquantenzahlen

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, \qquad j \in \tfrac{1}{2}\mathbb{N}$$

$$m = -j, -j+1, \dots, j-1, j$$

→ **Drehimpulseigenzustände**

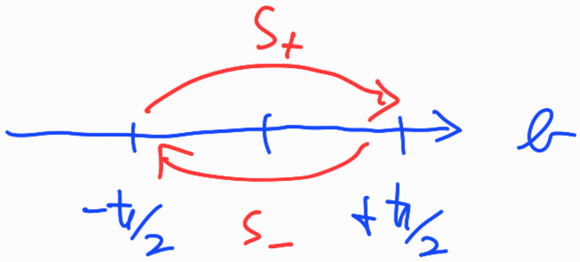
$$|j, m\rangle \qquad (\ |a, b\rangle\ )$$

$$J^2\,|j, m\rangle = \underline{\hbar^2 j(j+1)}\,|j, m\rangle \quad , \quad \underline{J_3}\,|j, m\rangle = \hbar\,\underline{m}\,|j, m\rangle$$

<u>Beispiele</u>: Spin des Elektrons:

$$S_3 \, |z\pm\rangle = \pm\frac{\hbar}{2} \, |z\pm\rangle$$

$\hookrightarrow \quad m = \pm 1/2 \, , \qquad \boxed{j \equiv s = 1/2 \quad !}$



$$\Big(\, |z+\rangle = |\underset{s}{1/2}, \underset{m}{1/2}\rangle \, , \quad |z-\rangle = |1/2, -1/2\rangle \Big)$$

$$S_+ = S_1 + i S_2 = \frac{\hbar}{2}(\sigma_1 + i\sigma_2) = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$S_- = S_1 - i S_2 = \qquad\qquad = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$S_+ \overset{|\downarrow\rangle}{\begin{pmatrix}0\\ 1\end{pmatrix}} = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \hbar\begin{pmatrix}1\\ 0\end{pmatrix} = \hbar \, |\uparrow\rangle$$

$$S_- \underset{|\uparrow\rangle}{\begin{pmatrix}1\\ 0\end{pmatrix}} = \qquad\qquad = \hbar \, |\downarrow\rangle$$

$$S_+ \begin{pmatrix}1\\ 0\end{pmatrix} = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = 0 \quad !$$

„Elektron ist Spin ½ Teilchen"