Infos zu Aufgaben 8 und 9: „Heisenberg-Bild"

Dynamik gemäß Schrödingergleichung:

$$\dot{\psi}(t) = -\frac{i}{\hbar} H \psi(t)$$

$$\Rightarrow \quad \psi(t) = U(t)\,\psi(0) \quad , \quad U(t) = e^{-\frac{i}{\hbar} H t}$$

$$\Rightarrow \quad \langle A \rangle_{\psi(t)} = \langle \psi(t) | A | \psi(t) \rangle$$

$$= \langle U(t)\psi(0) | A | U(t)\psi(0) \rangle$$

$$= \langle \psi(0) | U^{+}(t) A U(t) | \psi(0) \rangle$$

Heisenberg:

- Operatoren ($\hat{=}$ Observablen) zeitabhängig:

$$A_H(t) := U^{+}(t)\, \underbrace{A_H(0)}_{A}\, U(t)$$

- "Zustände" konstant:

$$\psi_0 = \psi(0)$$

„Heisenberg-Bild"

$$\rightarrow \boxed{\begin{aligned} \langle A \rangle_{\psi(t)} &= \langle A_H(t) \rangle_{\psi_0} \\ \dot{A}_H(t) &= \frac{i}{\hbar} [H, A_H(t)] \end{aligned}}$$

(vgl. Aufgabe 8)

Anwendung: Larmorpräzession (Aufgabe 9)

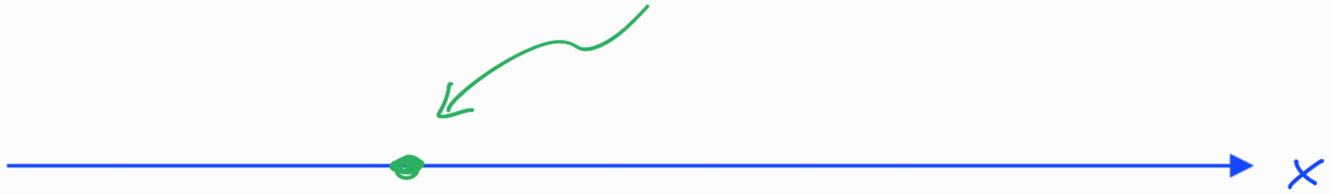
$$\vec{\mu} = \mu_0 \vec{\sigma} \xrightarrow{(*)} \vec{\mu}_H(t) = \mu_0 \vec{\sigma}_H(t)$$

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

$$(\sigma_i)_H(t) = \, ?$$

$$(*) \quad H = -\frac{B\mu_0}{\hbar} \sigma_3$$

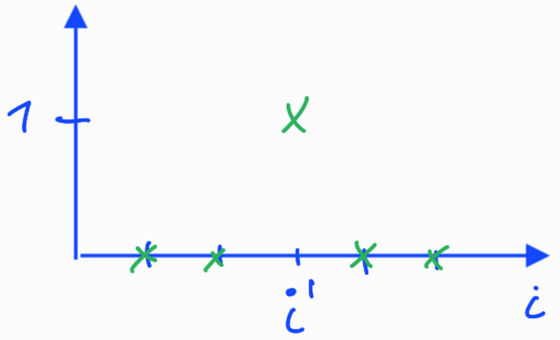
Letzte Vorlesung: QM des Punktteilchens (1D)

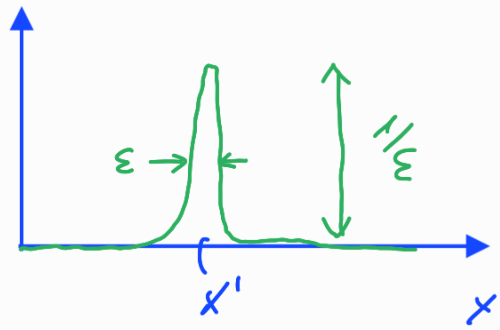


Ort x $\hat{=}$ hermit. Operator $\hat{x}$ mit kontinuierlichem Spektrum

$\rightarrow$ "Kontinuumsübergang" !

- $\delta_{ii'}$ $\rightarrow$ $\delta(x-x')$





- $\sum_i \delta_{ii'} = 1$ — $\int dx\, \delta(x-x') = 1$
- $\delta_{ii'} = 0 \;:\; i \neq i'$ — $\delta(x-x') = 0 \;:\; x \neq x'$

- orthn. Eigenbasis eines herm. Op $\hat{A}$:

$$\{|\varphi_1\rangle, \dots, |\varphi_n\rangle\}$$

- orth. Eigensystem des Ortsop. $\hat{x}$:

$$\{|\varphi_x\rangle\}_{x\in\mathbb{R}}$$

Orthonormalität

$$\langle\varphi_i|\varphi_{i'}\rangle = \delta_{ii'} \quad \longrightarrow \quad \langle\varphi_x|\varphi_{x'}\rangle = \delta(x-x')$$

Vollständigkeit

$$\mathbb{1}_{\mathcal{H}} = \sum_i |\varphi_i\rangle\langle\varphi_i| \quad \longrightarrow \quad \mathbb{1}_{\mathcal{H}} = \int dx\, |\varphi_x\rangle\langle\varphi_x|$$

Spektraldarstellung

$$\hat{A} = \mathbb{1}\,\hat{A}\,\mathbb{1} \quad \longrightarrow \quad \hat{x} = \mathbb{1}\,\hat{x}\,\mathbb{1}$$

$$= \sum_{ii'} |\varphi_i\rangle\underbrace{\langle\varphi_i|\overbrace{\hat{A}|\varphi_{i'}\rangle}^{a_{i'}|\varphi_{i'}\rangle}}_{a_{i'}\delta_{ii'}}\langle\varphi_{i'}| \qquad = \int dx\int dx'\, |\varphi_x\rangle\underbrace{\langle\varphi_x|\overbrace{\hat{x}|\varphi_{x'}\rangle}^{x'|\varphi_{x'}\rangle}}_{x'\delta(x-x')}\langle\varphi_{x'}|$$

$$= \sum_i a_i |\varphi_i\rangle\langle\varphi_i| \qquad = \int dx\, x\, |\varphi_x\rangle\langle\varphi_x|$$

Komponentendarst.

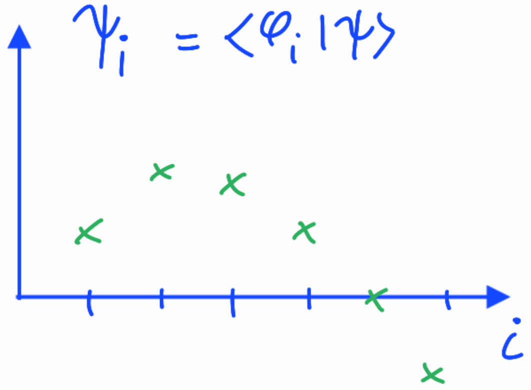
$$\mathcal{H} \ni |\psi\rangle = \mathbb{1}\,|\psi\rangle \qquad \longrightarrow \qquad |\psi\rangle = \mathbb{1}\,|\psi\rangle$$

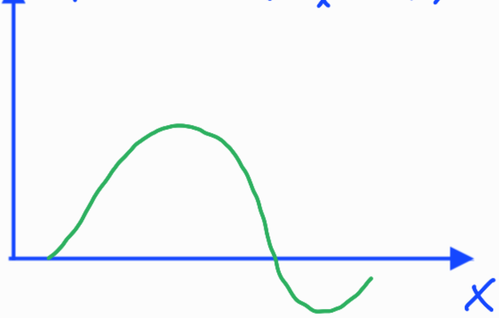
$$= \sum_i |\varphi_i\rangle \underbrace{\langle\varphi_i|\psi\rangle}_{\psi_i} \qquad\qquad = \int dx\, |\varphi_x\rangle \underbrace{\langle\varphi_x|\psi\rangle}_{\psi(x)}$$

$$\rightarrow \quad |\psi\rangle = \sum_i \psi_i\, |\varphi_i\rangle \qquad\qquad \rightarrow \quad |\psi\rangle = \int dx\, \psi(x)\, |\varphi_x\rangle$$

Komponenten von $|\psi\rangle$ — Wellenfunktion von $|\psi\rangle$

$$\psi_i = \langle\varphi_i|\psi\rangle \qquad \longrightarrow \qquad \psi(x) = \langle\varphi_x|\psi\rangle$$





Skalarprodukt

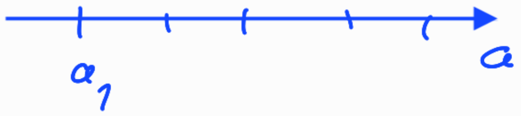
$$\langle\psi|\chi\rangle = \langle\psi|\mathbb{1}|\chi\rangle \qquad \longrightarrow \qquad \langle\psi|\chi\rangle = \langle\psi|\mathbb{1}|\chi\rangle$$

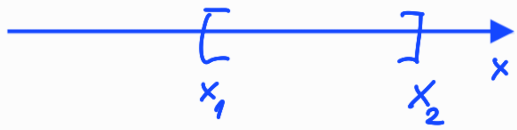
$$= \langle\psi|\Big(\sum_i |\varphi_i\rangle\langle\varphi_i|\Big)|\chi\rangle \qquad\qquad = \langle\psi|\Big(\int dx\, |\varphi_x\rangle\langle\varphi_x|\Big)|\chi\rangle$$

$$= \sum_i \psi_i^*\, \chi_i \qquad\qquad = \int dx\, \psi^*(x)\, \chi(x)$$

Bornsche Regel

Zust. $|\psi\rangle$ $\rightarrow$ $|\psi\rangle$





Wkt., dass Messwert $a \in \{a_{i_0}, \ldots, a_{i_0+m}\}$:

$$P = \sum_{i=i_0}^{i_0+m} |\langle \varphi_i | \psi \rangle|^2 = \sum_{i=i_0}^{i_0+m} |\psi_i|^2$$

Wkt., dass Teilchen-Ort $x \in [x_1, x_2]$:

$$P = \int_{x_1}^{x_2} dx\, |\langle \varphi_x | \psi \rangle|^2 = \int_{x_1}^{x_2} dx\, |\psi(x)|^2$$

d.h. $|\psi(x)|^2 =$ <u>Aufenthaltswkts.-dichte</u>

Erwartungswerte

$$\langle A \rangle_\psi = \sum_i a_i p_i = \sum_i a_i |\psi_i|^2 = \langle \psi | A | \psi \rangle$$

$\rightarrow$

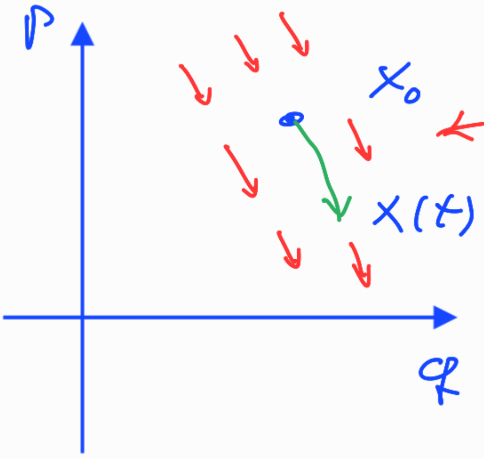
$$\langle \hat{x} \rangle_\psi = \int dx\, x\, |\langle \varphi_x | \psi \rangle|^2 = \int dx\, x\, |\psi(x)|^2 = \langle \psi | \hat{x} | \psi \rangle$$

Ort: $\hat{x}$, $\{|\varphi_x\rangle\}$, ✓

Impuls p des phys. Teilchens ?

↳ herm. Op. $\hat{p}$, $\{|\tilde{e}_p\rangle\}$, ... ?

a) Hamiltonfkt. $H(q,p)$: Erzeuger der Zeitentwicklung!

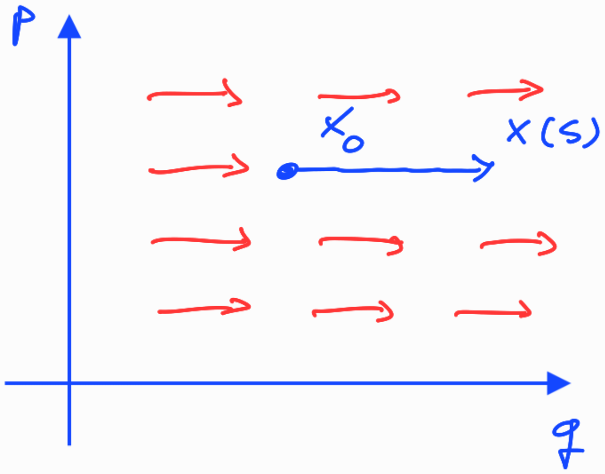


$$V_H = \left(\frac{\partial H}{\partial p} , - \frac{\partial H}{\partial q} \right)$$

↳ $\dot{x}(t) \overset{!}{=} V_H(x(t))$

↳ $x_0 \xrightarrow{t} x(t)$

b) Impuls p : Erzeuger der Translation!



$x_0 = (q_0, p_0) \longrightarrow$

$x(s) = (q_0 + s, p_0)$

(("$H \overset{!}{=} p$"
$t \rightarrow s$))

$$V_p = \left(\underbrace{\frac{\partial p}{\partial p}}_{1} , \underbrace{-\frac{\partial p}{\partial q}}_{0} \right) = (1, 0)$$

$$\rightsquigarrow \quad x'(s) \overset{!}{=} V_p(x(s)) = (1,0) \quad !$$

$$\Leftrightarrow \quad \begin{pmatrix} q'(s) \\ p'(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \quad x_0 = (q_0, p_0) \longrightarrow \underline{\underline{X(s) = (q_0 + s, p_0)}}$$

<u>in der QM soll es genauso sein!</u>

d.h. <u>Hamiltonop.</u> H : Erzeuger des <u>Zeitentwicklungsoperators</u>

$$U(t) = e^{-\frac{i}{\hbar} H t}$$

d.h. $$\boxed{H = i\hbar \frac{d}{dt} U(t) \Big|_{t=0}}$$

analog mit Translationen und Impuls $\longrightarrow$

b) **Def.:** Impulsoperator $\hat{p}$ ist Erzeuger der Translationen

↳ Translationsoperator

$$T(s) : |\varphi_x\rangle \longmapsto |\varphi_{x+s}\rangle$$

- $T(s_1)\,T(s_2) = T(s_1+s_2)$
- $T(s)$ unitär!

$$\Rightarrow \quad \boxed{p := i\hbar \frac{d}{ds} T(s)\Big|_{s=0}}$$

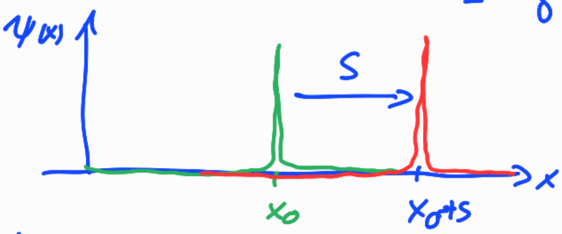
Wirkung von p auf Wellenfkt?

$$\begin{tikzcd}
\text{Zustand:} & |\psi\rangle \arrow[r, "p"] \arrow[d] & p|\psi\rangle \\
\text{Wellenfkt:} & \psi(x) \arrow[r, "-i\hbar\frac{\partial}{\partial x}"] & -i\hbar\frac{\partial}{\partial x}\psi(x)
\end{tikzcd}$$

$|\psi\rangle = |\varphi_{x_0}\rangle$

Wellefkt? $\quad \psi(x) = \langle \varphi_x | \varphi_{x_0}\rangle$

$= \delta(x-x_0)$



T(s):

$|\psi\rangle \to T(s)|\psi\rangle = T(s)|\varphi_{x_0}\rangle$

$= |\varphi_{x_0+s}\rangle = |\psi'\rangle$

$\hookrightarrow \psi'(x) = \langle\varphi_x|\varphi_{x_0+s}\rangle$

$= \delta(x-(x_0+s))$

• Zustand $|\chi\rangle$ mit Wellenfkt. $\chi(x)$:

$\hookrightarrow |\chi\rangle = \int dx\, \chi(x)|\varphi_x\rangle$

$|\psi\rangle = \int dx\,\psi(x)|\varphi_x\rangle \qquad (*)$

$P|\psi\rangle \overset{\text{Def.}}{=} i\hbar \frac{d}{ds} T(s)|\psi\rangle\Big|_{s=0}$

$\overset{(*)}{=} i\hbar\frac{d}{ds}\int dx\,\psi(x)|\varphi_{x+s}\rangle\Big|_{s=0}$

$x \to x-s$

$= i\hbar \frac{d}{ds}\int dx\,\psi(x-s)|\varphi_x\rangle\Big|_{s=0}$

$= \int dx \boxed{\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)}|\varphi_x\rangle$

Wellenfkt. von $P|\psi\rangle$

weitere Eigenschaften

(i) $\quad p = p^{+}$

(ii) $\quad [x, p] = i\hbar \mathbb{1}$

(iii) $$T(s) = e^{-\frac{i}{\hbar}\hat{p} s} \qquad !$$

zu (iii):

$$T'(s) = \frac{d}{da} T(s+a)\Big|_{a=0}$$

$$= \underbrace{\frac{d}{da} T(a)\Big|_{a=0}}_{\text{def.} \; =} T(s)$$

$$= \overbrace{-\frac{i}{\hbar}\hat{p}}\; T(s)$$

$T(s)$ genügt DGL: $\quad T'(s) = -\frac{i}{\hbar}\hat{p}\, T(s)$

zur A.W. $\quad T(0) = \mathbb{1}$

$$\longrightarrow \quad T(s) = e^{-\frac{i}{\hbar}\hat{p}\cdot s} \qquad \boxtimes$$

zu (i): $\quad \mathbb{1} \overset{!}{=} T^\dagger(s)\, T(s)$

$$= \left(e^{-\frac{i}{\hbar}\hat{p}s}\right)^\dagger \cdot e^{-\frac{i}{\hbar}\hat{p}s}$$

$\uparrow$ (iii)

$$= \left(e^{+\frac{i}{\hbar}\hat{p}^\dagger s} \cdot e^{-\frac{i}{\hbar}\hat{p}s} \right)$$

$$= e^{+\frac{i}{\hbar}\underbrace{(\hat{p}^\dagger - \hat{p})}_{\overset{!}{=}\, 0}s} \overset{!}{=} \mathbb{1} \; !$$

$$\rightarrow \hat{p}^\dagger \overset{!}{=} \hat{p}$$

(ii) $\quad [\hat{x}, \hat{p}] \overset{!}{=} i\hbar \mathbb{1} \quad !$

$$[\hat{x}, T(s)] = s\,T(s) \quad ! \quad (*)$$

- $\hat{x}\, T(s) |\varphi_x\rangle = (x+s)\, T(s) |\varphi_x\rangle$
- $T(s)\, \hat{x} |\varphi_x\rangle = x\, T(s) |\varphi_x\rangle$

$$[\hat{x}, T(s)] |\varphi_x\rangle = s\, T(s) |\varphi_x\rangle$$

$$[\hat{x}, \hat{p}] = i\hbar \frac{d}{ds} [\hat{x}, T(s)]\Big|_{s=0} = i\hbar \frac{d}{ds}(s\,T(s))\Big|_{s=0}$$

$$= i\hbar\, (\underbrace{T(0)}_{= \mathbb{1}} + 0\, \cancel{T'(s)}) = i\hbar\, \mathbb{1}. \; !$$

Impulseigenzustände $|\tilde{\varphi}_p\rangle$

($\Rightarrow$ $\langle\tilde{\varphi}_p|\tilde{\varphi}_{p'}\rangle = ?$, $\int dp\, |\tilde{\varphi}_p\rangle\langle\tilde{\varphi}_p|$)

Impulseigenzust. $|\tilde{\varphi}_p\rangle$ zum Impulseigenwert p

besitzt Wellenfkt.:

$$\boxed{\tilde{\varphi}_p(x) = e^{i\frac{p}{\hbar}x}} \quad !$$

ebene Welle mit Wellenzahl

$k = p/\hbar$

$$\hat{p}\,|\tilde{\varphi}_p\rangle \overset{!}{=} p\,|\tilde{\varphi}_p\rangle$$

$$\Leftrightarrow \quad -i\hbar\frac{\partial}{\partial x}\tilde{\varphi}_p(x) \overset{!}{=} p\,\tilde{\varphi}_p(x) \quad \checkmark$$

denn: $$-i\hbar\frac{\partial}{\partial x}e^{i\frac{p}{\hbar}x} = p\,e^{ipx/\hbar} \quad \checkmark$$

Orthonormalität:

$$\langle \tilde{\varphi}_p | \tilde{\varphi}_{p'} \rangle = \int dx \; e^{-i\frac{p}{\hbar}x} \cdot e^{+i\frac{p'}{\hbar}x}$$

$$= \int dx \; e^{i\frac{(p'-p)}{\hbar}x}$$

$$= 2\pi \, \delta\left(\frac{p'-p}{\hbar}\right) = 2\pi\hbar \, \delta(p'-p) \; !$$