Lösungshinweise Blatt 10

38) $Z(\beta) = 1 + e^{-\beta\varepsilon}$

$$\rightarrow \quad n_1 = \frac{e^{-\beta\varepsilon}}{Z(\beta)} = \frac{1}{e^{\beta\varepsilon}+1} = \frac{1}{e^{\varepsilon/kT}+1}$$

$$n_0 = \frac{e^{-\beta\cdot 0}}{Z(\beta)} = \frac{1}{1+e^{-\beta\varepsilon}} = \frac{1}{1+e^{-\varepsilon/kT}} \xrightarrow[T\to 0]{} 1$$

d.h. für $T=0$ System im Grundzustand

- $n_1/n_0 = e^{-\beta\varepsilon} = e^{-\varepsilon/kT} < 1$

d.h. Grundzustand für alle $T<\infty$ wahrscheinlichsten Zustand,

- für $\beta \to 0$, d.h. $T\to\infty$: $n_1 = n_0 = \frac{1}{2}$.

39) $H(x,p) = p^2/2m + \frac{m}{2}\omega^2 x^2$

$$\rightarrow \quad Z(\beta) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \; e^{-\beta H(x,p)}$$

$$= \int_{-\infty}^{\infty} dx \; e^{-\beta\frac{m\omega^2}{2}x^2} \int_{-\infty}^{\infty} dp \; e^{-\beta p^2/2m}$$

$$= \left(\frac{2\pi}{\beta m \omega^2}\right)^{1/2} \left(\frac{2\pi m}{\beta}\right)^{1/2} = \frac{2\pi}{\omega\beta}$$

$$\int dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\rightsquigarrow \langle E \rangle_T = -\frac{\partial}{\partial\beta} \ln\left(\frac{2\pi}{\omega\beta}\right) = \frac{1}{\beta} = kT \,.$$

40)

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n$$

$$= \left(1 - e^{-\beta\hbar\omega}\right)^{-1}$$

$$\rightsquigarrow \langle E \rangle_T = -\frac{\partial}{\partial\beta} \ln Z(\beta) = \frac{\partial}{\partial\beta} \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$= \frac{\hbar\omega\, e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

$$= \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \begin{cases} \simeq \hbar\omega\, e^{-\hbar\omega/kT} & : kT \ll \hbar\omega \\ \simeq kT & : kT \gg \hbar\omega \end{cases}$$

$$e^{\hbar\omega/kT} \simeq 1 + \frac{\hbar\omega}{kT} + O\left(\frac{\hbar\omega}{kT}\right)^2$$