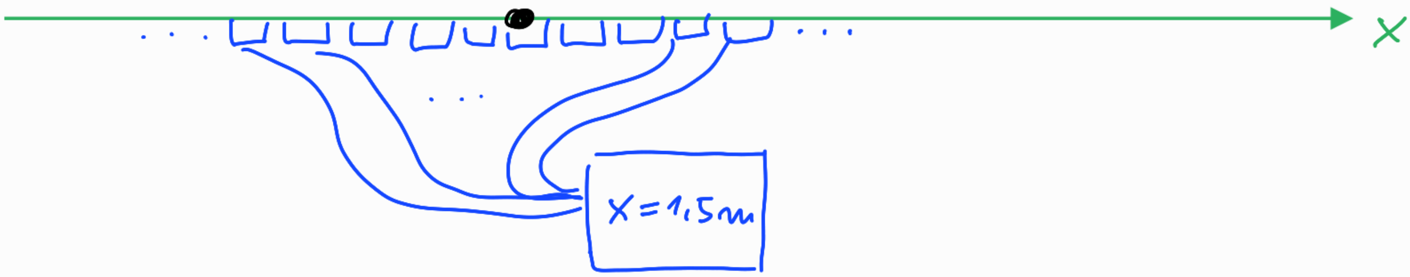
Quantenmechanik des Punktteilchens



Ortskoordinate x = Observable mit kontinuierlichen Messwerten $x \in \mathbb{R}$

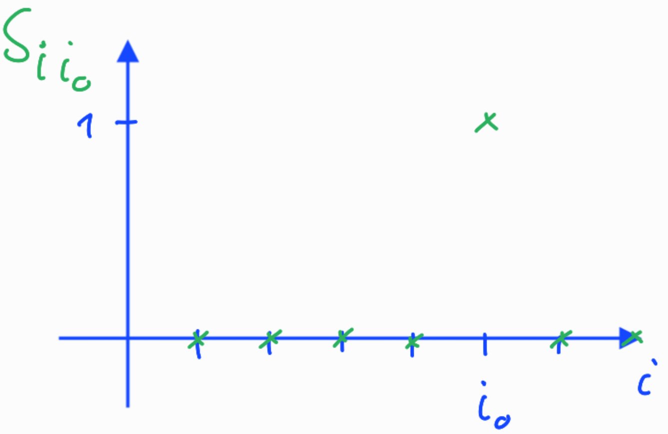
$\hat{=}$ herm. Op. X mit Kontinuum an Eigenwerten $x \in \mathbb{R}$ zu orthonormalen Eigenzuständen $\{|\varphi_x\rangle\}_{x \in \mathbb{R}}$

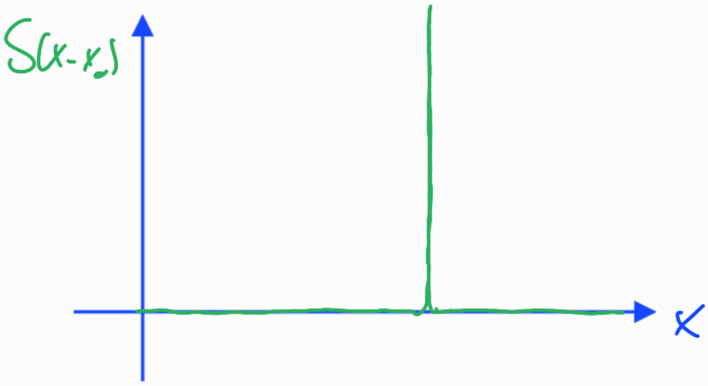
diskret	kontinuierlich
$B = \{ \|\varphi_i\rangle \}_{i=1,2,3,\dots}$	$B = \{ \|\varphi_x\rangle \}_{x \in \mathbb{R}}$

Orthonormalität

$\langle \varphi_i \| \varphi_{i_0} \rangle = \delta_{i i_0}$	$\langle \varphi_x \| \varphi_{x_0} \rangle \overset{!}{=} \delta(x - x_0)$

Dirac-Deltafunktion





- $\delta_{ii_0} = 0$ für $i \neq i_0$
- $\sum_i \delta_{ii_0} = 1$
- $\sum_i a_i \, \delta_{ii_0} = a_{i_0}$

- $\delta(x-x_0) = 0$ für $x \neq x_0$
- $\int_{-\infty}^{\infty} dx \, \delta(x-x_0) \overset{!}{=} 1$
- $\int dx \, f(x) \, \delta(x-x_0) = f(x_0)$

Vollständigkeit

$$\boxed{\sum_i |\varphi_i\rangle\langle\varphi_i| \overset{!}{=} \mathbb{1}}$$

$$\hookrightarrow \quad |\psi\rangle = \mathbb{1} \, |\psi\rangle$$

$$= \sum_i |\varphi_i\rangle \underbrace{\langle\varphi_i|\psi\rangle}_{\psi_i}$$

i-te Komponente von $|\psi\rangle$ bzgl. $\mathcal{B}$:

$$\psi_i = \langle\varphi_i|\psi\rangle$$

$$\hookrightarrow \quad |\psi\rangle = \sum_i \psi_i \, |\varphi_i\rangle$$

$$\boxed{\int dx \, |\varphi_x\rangle\langle\varphi_x| \overset{!}{=} \mathbb{1}}$$

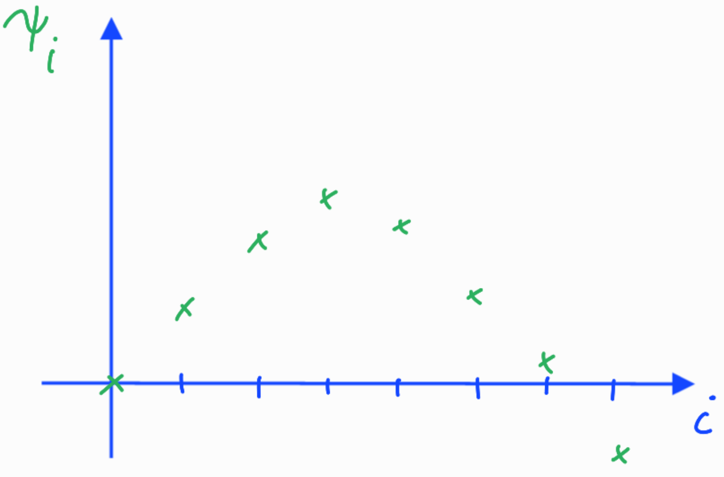
$$\hookrightarrow \quad |\psi\rangle = \mathbb{1} \, |\psi\rangle$$

$$= \int dx \, |\varphi_x\rangle \underbrace{\langle\varphi_x|\psi\rangle}_{\psi(x)}$$

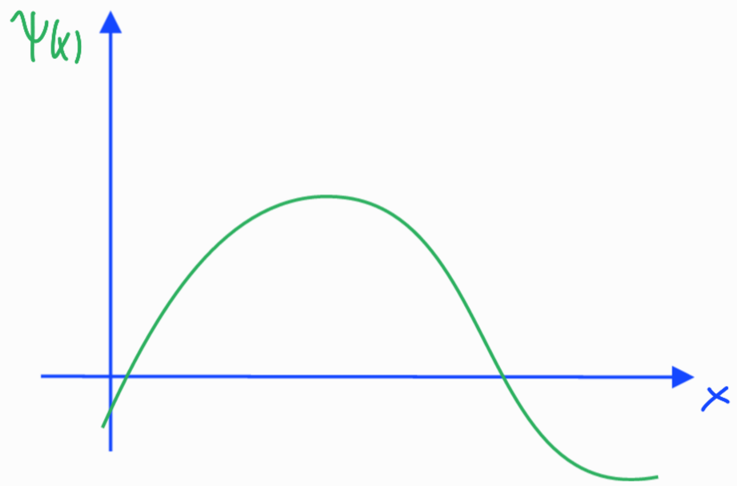
Wellenfunktion von $|\psi\rangle$ an der Stelle x

$$\psi(x) = \langle\varphi_x|\psi\rangle$$

$$\hookrightarrow \quad |\psi\rangle = \int dx \, \psi(x) \, |\varphi_x\rangle$$



$$|\psi\rangle = \sum_i \psi_i \, |\varphi_i\rangle$$



$$|\psi\rangle = \int dx \, \psi(x) \, |\varphi_x\rangle$$

Skalarprodukt

$$\langle \psi | \chi \rangle = \langle \psi | \mathbb{1} | \chi \rangle$$

$$= \langle \psi | \underbrace{\Big(\sum_i |\varphi_i\rangle\langle\varphi_i| \Big)}_{= \mathbb{1}} | \chi \rangle$$

$$= \sum_i \underbrace{\langle \psi | \varphi_i \rangle}_{\psi_i^*} \underbrace{\langle \varphi_i | \chi \rangle}_{\chi_i}$$

$$= \sum_i \psi_i^* \chi_i$$

$$\langle \psi | \chi \rangle = \langle \psi | \mathbb{1} | \chi \rangle$$

$$= \langle \psi | \underbrace{\Big(\int_{-\infty}^{\infty} dx \, |\varphi_x\rangle\langle\varphi_x| \Big)}_{= \mathbb{1}} | \chi \rangle$$

$$= \int_{-\infty}^{\infty} dx \, \underbrace{\langle \psi | \varphi_x \rangle}_{\psi^*(x)} \underbrace{\langle \varphi_x | \chi \rangle}_{\chi(x)}$$

$$\boxed{\langle \psi | \chi \rangle = \int_{-\infty}^{\infty} dx \, \psi^*(x) \, \chi(x)}$$

Norm

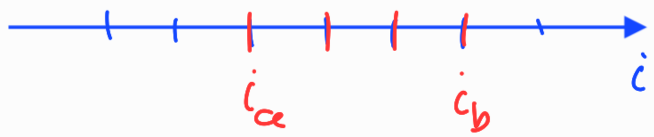
$$|\psi|^2 = \langle\psi|\psi\rangle$$

$$= \sum_i |\psi_i|^2$$

$$|\psi|^2 := \langle\psi|\psi\rangle$$

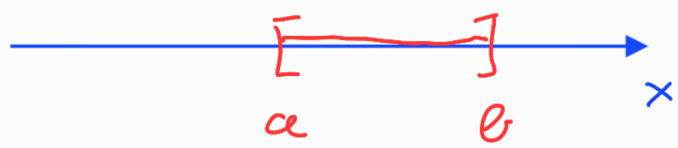
$$= \int dx\, |\psi(x)|^2$$

Bornsche Regel



Wkt., dass Messwert $i_a, i_{a+1}, \dots,$ oder i_b gemessen wird:

$$p = \sum_{i=i_a}^{i_b} \underbrace{|\langle\varphi_i|\psi\rangle|^2}_{p_i}$$



Wkt., das Ortsmessung $x \in [a, b]$

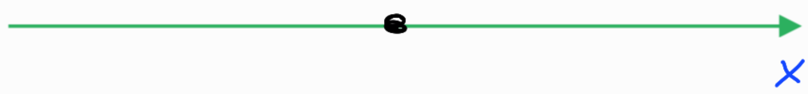
$$p = \int_a^b dx\, |\langle\varphi_x|\psi\rangle|^2$$

Aufenthaltswahrscheinlichkeitsdichte:

$$\rho(x) = |\langle\varphi_x|\psi\rangle|^2$$

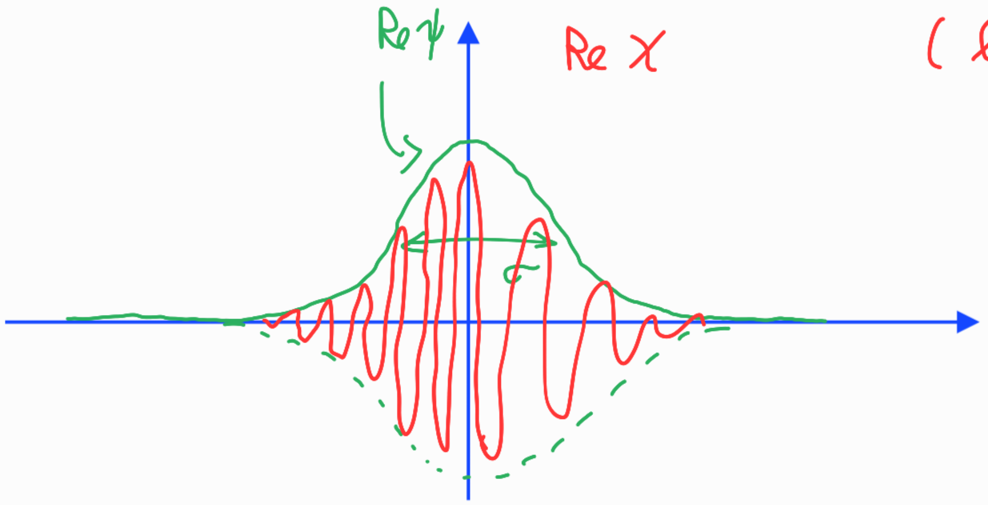
Beispiele zum Formalismus:

Teilchen in 1D



1) Teilchen im Zustand $|\psi\rangle$ mit Wellenfunktion

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \, e^{-x^2/4\sigma^2}$$



$(\ell \gg \sigma)$

Normierung: $|\psi|^2 \overset{?}{=} 1$

$$\Leftrightarrow \quad 1 = \int dx \, |\psi(x)|^2 = \int_{-\infty}^{\infty} dx \, \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\frac{\pi}{1/2\sigma^2}} = 1$$

$\mathrm{Re}\, a > 0$

$a, b \in \mathbb{C}$ (*)

$$\boxed{\text{Gauß-Integral:} \quad \int_{-\infty}^{\infty} dx \, e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} \, e^{b^2/4a}}$$

2) Zustand $|\chi\rangle$ mit Wellenfunktion

$$\chi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\, e^{-x^2/4\sigma^2} \cdot e^{ikx}$$

$$\mathrm{Re}\,\chi(x) = \quad '' \qquad '' \quad \cdot \cos(kx) \qquad k \in \mathbb{R}$$

↳ Normierung: $\int dx\, |\chi(x)|^2 = \int dx\, |\psi(x)|^2 = 1$ ✓

3)

$$\langle \psi | \chi \rangle = \int dx\, \psi^*(x)\, \chi(x)$$

$$= \int dx \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\underbrace{x^2/2\sigma^2}_{a = 1/2\sigma^2} + \underbrace{ik}_{e}x}$$

$$\overset{(*)}{=} \frac{1}{\cancel{\sqrt{2\pi\sigma^2}}} \sqrt{\frac{\cancel{\pi}}{\cancel{1/2\sigma^2}}}\, e^{-\frac{k^2\sigma^2}{2}}$$

∘ $\langle \psi | \chi \rangle = e^{-\frac{k^2\sigma^2}{2}}$

∘ $|\psi(x)|^2 = \dfrac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} = |\chi(x)|^2$

3) Ortsoperator $\hat{X}$

$$A = \sum_i a_i |\varphi_i\rangle\langle\varphi_i|$$

Eigenwerte

Eigenzust.

Test: $A|\varphi_k\rangle = \sum_i a_i |\varphi_i\rangle \underbrace{\langle\varphi_i|\varphi_k\rangle}_{\delta_{ik}} = a_k |\varphi_k\rangle$ ✓

$\to x \in \mathbb{R}$, $\{|\varphi_x\rangle\}_{x\in\mathbb{R}}$

$$\boxed{\hat{X} = \int_{-\infty}^{+\infty} dx \; x \, |\varphi_x\rangle\langle\varphi_x|}$$

Test: $\hat{X}|\varphi_{x_0}\rangle \stackrel{?}{=} x_0 |\varphi_{x_0}\rangle$ ✓

$$\left(\left(\int_{-\infty}^{\infty} dx \; x \, |\varphi_x\rangle \underbrace{\langle\varphi_x|\varphi_{x_0}\rangle}_{\delta(x-x_0)} \right.\right.$$

$$= \int_{-\infty}^{\infty} dx \; x \, |\varphi_x\rangle \, \delta(x-x_0) = x_0 |\varphi_{x_0}\rangle \; ✓$$

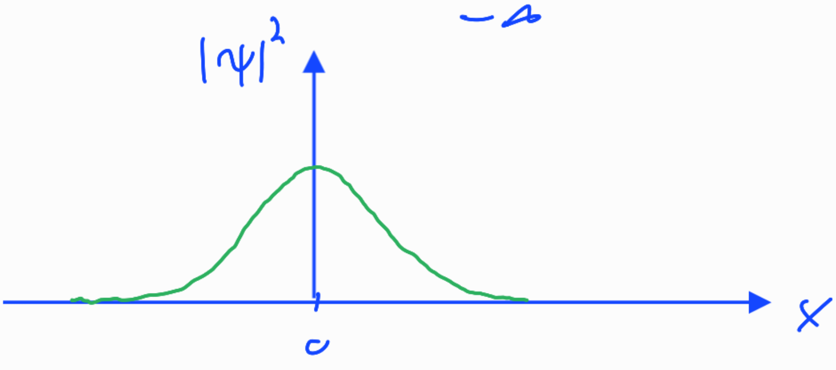
4) Erwartungswert von X im Zust. $|\psi\rangle$:

$$
\begin{aligned}
\langle X \rangle_\psi &= \langle \psi | \hat{X} | \psi \rangle \\
&\underset{3)}{=} \langle \psi | \int_{-\infty}^{\infty} dx \; x \, |\varphi_x\rangle \langle \varphi_x | \psi \rangle \\
&= \int_{-\infty}^{\infty} dx \; x \, \underbrace{\underbrace{\langle \psi | \varphi_x \rangle}_{\psi^*(x)} \underbrace{\langle \varphi_x | \psi \rangle}_{\psi(x)}}_{\|}
\end{aligned}
$$

$$
\boxed{\langle x \rangle_\psi = \int_{-\infty}^{\infty} dx \; x \, \underbrace{|\psi(x)|^2}_{\rho(x)}}
$$

Für $\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} e^{-x^2/4\sigma^2}$

$$
\hookrightarrow \langle x \rangle_\psi = \int_{-\infty}^{\infty} dx \; x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} = 0
$$



5) Erwartungswert von x^2:

$$\rightarrow \langle x^2 \rangle_{|\psi\rangle} = \int_{-\infty}^{\infty} dx \; x^2 \; |\psi(x)|^2$$

alternativ:

$$\hat{x} = \int_{-\infty}^{\infty} dx \; x \; |\varphi_x\rangle\langle\varphi_x|$$

$$\hat{x}^2 = \int_{-\infty}^{\infty} dx \; x^2 \; |\varphi_x\rangle\langle\varphi_x|$$

$\vdots$

$$\hat{x}^n = \int_{-\infty}^{\infty} dx \; x^n \; |\varphi_x\rangle\langle\varphi_x|$$

$$f(\hat{x}) := \int_{-\infty}^{\infty} dx \; f(x) \; |\varphi_x\rangle\langle\varphi_x|$$

$$\hookrightarrow \langle \hat{x} \rangle_{|\psi\rangle} = \langle\psi| \int_{-\infty}^{\infty} dx \; x^2 \; |\varphi_x\rangle\langle\varphi_x| \psi\rangle$$

$$= \int_{-\infty}^{+\infty} dx \; x^2 \; |\psi(x)|^2$$

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} e^{-x^2/4\sigma^2}$$

$$\Rightarrow \underline{\langle x^2 \rangle_\psi} = \int_{-\infty}^{+\infty} dx\, x^2 \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx\, x^2 e^{-\left(\frac{x}{\sqrt{2\sigma^2}}\right)^2}$$

$$= \frac{\sqrt{2\sigma^2}^3}{\sqrt{2\pi\sigma^2}} \underbrace{\int_{-\infty}^{\infty} dy\, y^2 e^{-y^2}}_{= \frac{\sqrt{\pi}}{2}} = \underline{\sigma^2}$$

subst. $x = \sqrt{2\sigma^2}\, y$

$$\left(= -\frac{d}{d\lambda} \Big(\underbrace{\int_{-\infty}^{\infty} dy\, e^{-y^2\lambda}}_{= \sqrt{\frac{\pi}{\lambda}}} \Big) \Big|_{\lambda=1} = \frac{\sqrt{\pi}}{2\lambda^{3/2}} \Big|_{\lambda=1} = \frac{\sqrt{\pi}}{2} \right)$$