Letzte Woche:

<u>Ortsoperator</u> $\hat{x}$, $|\varphi_x\rangle$, $x \in \mathbb{R}$

d.h.
- $\hat{x}|\varphi_x\rangle = x|\varphi_x\rangle$
- $\hat{x} = \int dx \; x \; |\varphi_x\rangle\langle\varphi_x|$
- $\mathbb{1} = \int dx \; |\varphi_x\rangle\langle\varphi_x|$
- $\langle\varphi_x|\varphi_{x'}\rangle = \delta(x-x')$ ....

<u>heute</u>:

Impuls $\longrightarrow$ <u>Impulsoperator</u>

$\hat{p}$, $|\chi_p\rangle$, $p \in \mathbb{R}$

$$\hat{p} = ?$$

Nachtrag: $\hat{X} = \int dx \; x \; |\varphi_x\rangle\langle\varphi_x|$

$\hat{X}$ in Ortsdarstellung:

$$\boxed{\begin{tikzcd}[column sep=large]
\text{Zustand} & |\psi\rangle \arrow[r, "\hat{X}"] \arrow[d] & |\tilde{\psi}\rangle = \hat{X}\,|\psi\rangle \arrow[d] \\
\text{Wellenfkt.} & \psi(x) \arrow[r, "x\cdot"] & \tilde{\psi}(x) = x\,\psi(x)
\end{tikzcd}}$$

Warum?

- $\psi(x) = \langle\varphi_x|\psi\rangle$
- $\tilde{\psi}(x_0) = \langle\varphi_{x_0}|\tilde{\psi}\rangle = \langle\varphi_{x_0}|\hat{X}|\psi\rangle$

$$= \langle\varphi_{x_0}| \underbrace{\int dx \; x \; |\varphi_x\rangle\langle\varphi_x|}_{=\hat{X}} |\psi\rangle$$

$$= \int dx \; x \; \underbrace{\langle\varphi_{x_0}|\varphi_x\rangle}_{\delta(x_0-x)} \underbrace{\langle\varphi_x|\psi\rangle}_{\psi(x)}$$

$$= x_0 \, \psi(x_0)$$

Impuls in der Quantenmechanik (eines Punktteilchens)

Impuls (Energie) in der klassischen Mech.

Impuls:

- $p = m v$

b) $\boxed{\text{• } p \text{: Erzeuger der Translation}}$

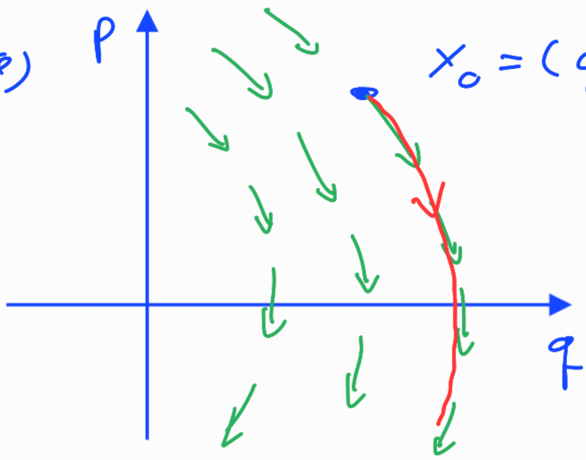
Energie

- $E = \frac{p^2}{2m} + V(q)$

  $\parallel$

a) $\boxed{\text{• } H \text{: Erzeuger der Zeitentwicklung (= Zeittranslation } t \to t + \tau \text{)}}$

$\rightarrow$ in der QM soll es genauso sein!

a) 

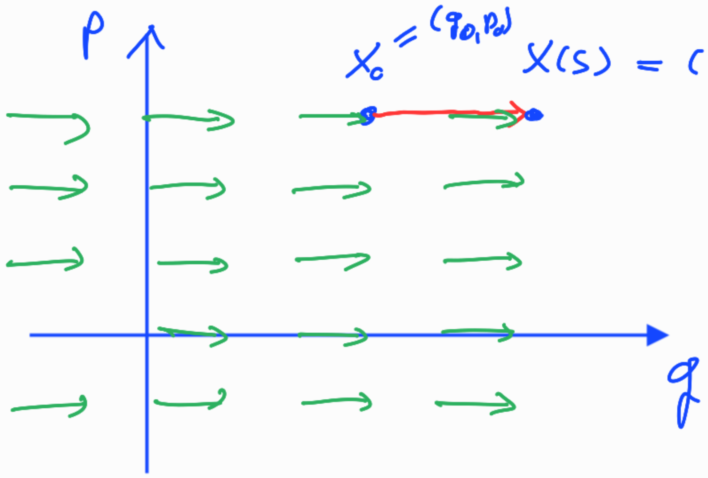
$x_0 = (q_0, p_0)$, Hamiltonfkt $H(p,q)$ $\left(= p^2/_{2m} + U(q)\right)$

$$V_H = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right)$$

$$\hookrightarrow \dot{x}(t) \overset{!}{=} V_H(x(t))$$

b) Translation: $T(s): q \mapsto q + s$

e) Translation: $T(s): \; q \longmapsto q+s$



$X(s) = (q_0 + s, p)$

$$V_p = (1, 0)$$

"$H = p$" !

$$V_p = \left(\frac{\partial p}{\partial p}, -\frac{\partial p}{\partial q} \right) \;\hat{=}$$

$$\dot{x}(s) = V_p(x(s))$$

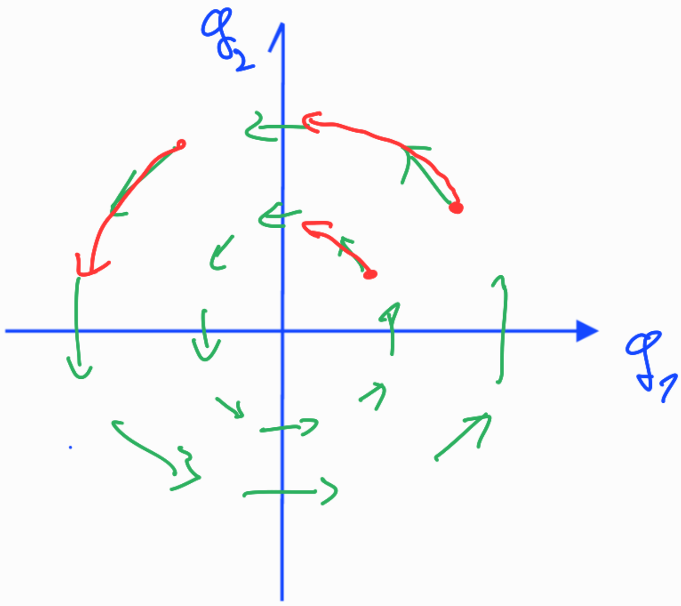
$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \quad x(s) = (q_0 + s, \; p_0) \overset{!}{=} T(s)(x_0)$$

f) Drehimpuls $\vec{L}$ = Erzeuger der Rotation

$L_3 =$ um $\vec{e}_3$-Achse!

$$L_3 = (\vec{q} \times \vec{p})_3 = q_1 p_2 - q_2 p_1$$



$$\downarrow$$

$$V_{L_3} = \left(\frac{\partial L_3}{\partial p_1}, \frac{\partial L_3}{\partial p_2}, -\frac{\partial L_3}{\partial q_1}, -\frac{\partial L_3}{\partial q_2} \right)$$

$$= \underline{\underline{(-q_2, +q_1, -p_2, +p_1)}}$$

.... in der QM soll es ebenso sein!

→ a) $\hat{H} \rightsquigarrow e^{-\frac{i}{\hbar}Ht} =: U(t)$ "Zeittranslation"

$$\left(|\psi_0\rangle \xrightarrow{t} |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi_0\rangle = U(t)|\psi_0\rangle \right)$$

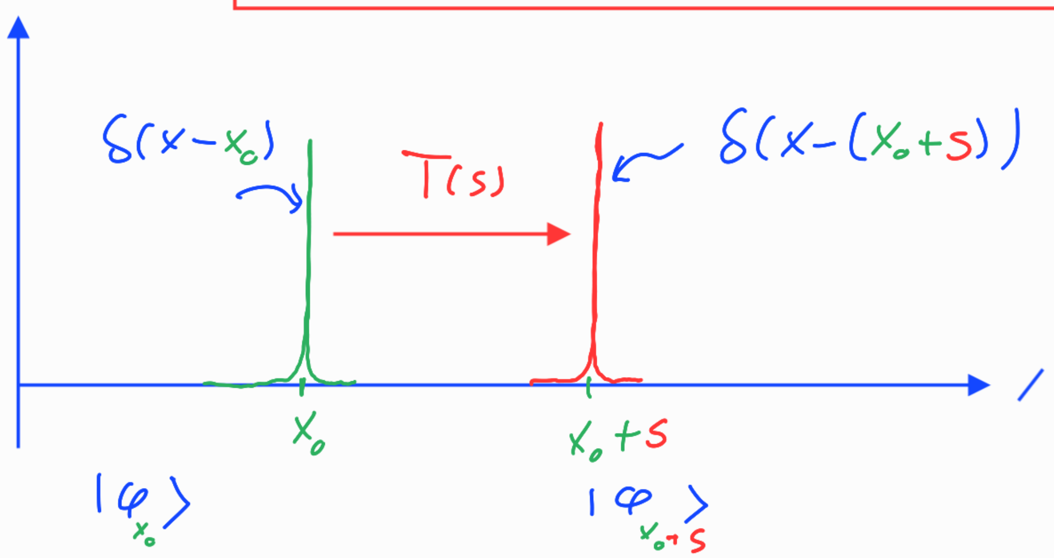
d.h. $$\boxed{\hat{H} = i\hbar \frac{d}{dt} U(t)\Big|_{t=0}}$$

b) $\hat{p} \rightsquigarrow \hat{T}(s)$: Translation um s

$$\hat{T}(s)\,|\varphi_x\rangle := |\varphi_{x+s}\rangle$$

→ Def. des Impulsop. :

$$\boxed{\hat{p} := i\hbar \frac{d}{ds} T(s)\Big|_{s=0}}$$



Impuls in Ortsdarstellung:

$$\boxed{\begin{array}{lccc} \text{Zust.} & |\psi\rangle & \xrightarrow{\hat{p}} & |\tilde{\psi}\rangle = \hat{p}\,|\psi\rangle \\ & \downarrow & & \\ \text{Wellenfkt} & \psi(x) & \xrightarrow{-i\hbar\,\partial/\partial x} & \tilde{\psi}(x) = -i\hbar \frac{\partial}{\partial x}\psi(x) \end{array}}$$

⌈ warum? • $\psi(x) = \langle \varphi_x | \psi \rangle$

$$? = \tilde{\psi}(x_0) = \langle \varphi_{x_0} | \tilde{\psi} \rangle = \langle \varphi_{x_0} | \hat{p} | \psi \rangle$$

$$\overset{\text{Def.}}{=} \langle \varphi_{x_0} | \, i\hbar \frac{d}{ds} T(s)\Big|_{s=0} | \psi \rangle$$

$$= i\hbar \frac{d}{ds} \langle \varphi_{x_0} | T(s) \overset{\mathbb{1}}{\vee} | \psi \rangle \Big|_{s=0}$$

$$= i\hbar \frac{d}{ds} \int dx \, \underbrace{\langle \varphi_{x_0} | \underbrace{T(s) | \varphi_x \rangle}_{|\varphi_{x+s}\rangle}}_{\delta(x_0 - (x+s)) = \delta(-x + x_0 - s)} \underbrace{\langle \varphi_x | \psi \rangle}_{\psi(x)} \Big|_{s=0}$$

$$\mathbb{1} = \int dx \, |\varphi_x\rangle\langle\varphi_x|$$

$$= i\hbar \frac{d}{ds} \psi(x_0 - s) \Big|_{s=0}$$

$$= \boxed{-i\hbar \frac{\partial}{\partial x}} \psi(x_0)$$

⌋

Eigenschaften des Impulsoperators:

(i) $[\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

(ii) $\hat{T}(a) = e^{-i\hat{p}a/\hbar}$ $\qquad (\, U(t) = e^{-iHt/\hbar}$

zu (i) $\qquad \hat{p} = i\hbar \frac{d}{dt} T(s)\Big|_{s=0}$

$$[\hat{x}, \hat{T}(s)] \overset{(*)}{=} s\,T(s) \quad \checkmark$$

$$C = \hat{x}\hat{T}(s) - \hat{T}(s)\hat{x}$$

- $\hat{x}\hat{T}(s)|\varphi_x\rangle = \hat{x}|\varphi_{x+s}\rangle = (x+s)|\varphi_{x+s}\rangle = (x+s)\underline{T(s)}|\varphi_x\rangle$
- $\hat{T}(s)\hat{x}|\varphi_x\rangle = \hat{T}(s)\,x\,|\varphi_x\rangle = \underline{x\,T(s)}|\varphi_x\rangle$

$$[\hat{x}, \hat{p}] = [\hat{x}, i\hbar \frac{d}{ds}\hat{T}(s)\Big|_{s=0}] = i\hbar \frac{d}{ds}[\hat{x}, \hat{T}(s)]\Big|_{s=0}$$

$$\overset{*}{=} i\hbar \frac{d}{ds}\left(s\,T(s)\right)\Big|_{s=0} = i\hbar \underset{\mathbb{1}}{\underset{\shortparallel}{\hat{T}(0)}} + i\hbar\, \underbrace{s\frac{d}{ds}T(s)\Big|_{s=0}}_{=0}$$

d. h. $\quad [\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

zu (ii): $\hat{T}(a) = e^{-i\hat{p}a/\hbar}$

DGL für $T(a)$:

$$\left.\frac{d T(a)}{da}\right|_a = \frac{d}{ds} \overbrace{T(a+s)}^{= T(a)T(s)}\Big|_{s=0}$$

$$= \underbrace{T(a) \cdot \frac{d}{ds} T(s)\Big|_{s=0}}_{= -\frac{i}{\hbar}\hat{p}}$$

Def: $\hat{p} = i\hbar \frac{d}{ds} T(s)\Big|_{s=0}$

$$\Rightarrow \quad \frac{d}{da} T(a) = -\frac{i}{\hbar}\hat{p}\, T(a)$$

$\Rightarrow$ Lsg zum AW $T(0) = \mathbb{1}$ ist

$$T(a) = e^{-i\frac{\hat{p}}{\hbar}a}\, \mathbb{1} \, .$$

Impulseigenzustände

Finde Zustand $|\chi_p\rangle$ so, dass

$$\hat{p}\,|\chi_p\rangle \overset{!}{=} p\,|\chi_p\rangle \,, \qquad p \in \mathbb{R}$$

$\langle\varphi_x|\cdots$

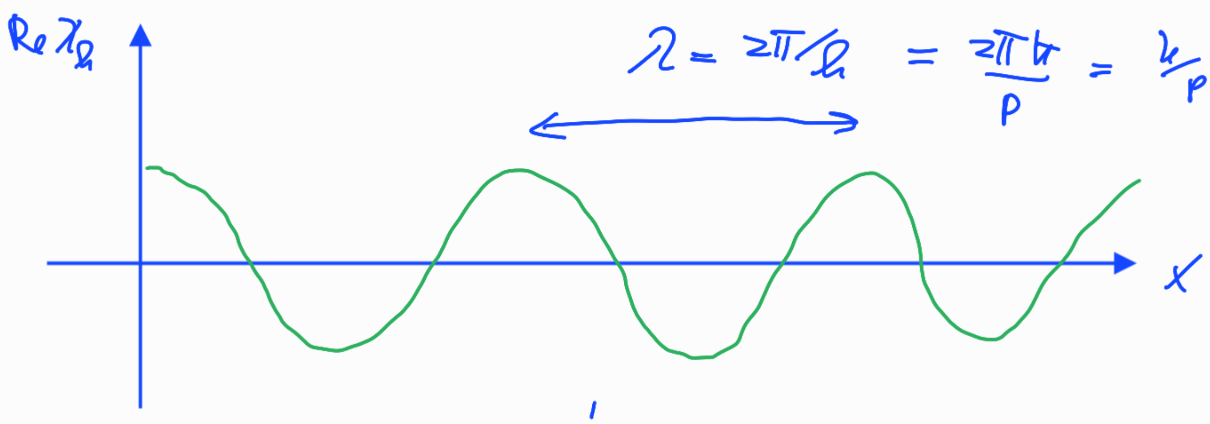
$$-i\hbar\frac{\partial}{\partial x}\chi_p(x) \overset{!}{=} p\,\chi_p(x)$$

$$\Leftrightarrow \qquad \chi_p'(x) = i\frac{p}{\hbar}\,\chi_p(x) \qquad (y' = a y)$$

Lösung zur AW $\chi_p(0) = \underline{\underline{1}}$ (Konvention!)

$$\boxed{\chi_k(x) \equiv \underset{\hbar k}{\chi_p}(x) = e^{i\frac{p}{\hbar}x} = e^{ikx}}$$

Wellenzahl $k = p/\hbar$



Impulseigenzustand $|\chi_k\rangle$ zum

Impulseigenwert $p = \hbar k$ besitzt Wellen

funktion

$$\chi_k(x) = e^{ikx}$$

$\Rightarrow$ Orthonormalität:

$\langle \varphi_x | \varphi_{x'} \rangle = \delta(x-x')$

- $$\boxed{\langle \chi_k | \chi_{k'} \rangle = 2\pi\, \delta(k-k')}$$

$\mathbb{1} = \int dx\, |\varphi_x\rangle\langle\varphi_x|$

- $$\boxed{\int \frac{dk}{2\pi} |\chi_k\rangle\langle\chi_k| \overset{!}{=} \mathbb{1}} = \int dx\, |\varphi_x\rangle\langle\varphi_x|$$