Impuls $p \overset{!}{=}$ Erzeuger der Translation

$$\hat{T}(s): \; |\varphi_x\rangle \longmapsto |\varphi_{x+s}\rangle$$

d.h. $$\boxed{\hat{p} := i\hbar \frac{d}{ds} T(s) \Big|_{s=0}}$$

$\rightsquigarrow$ p in Ortsdarstellung:

$$\psi(x) \overset{p}{\longmapsto} -i\hbar \frac{\partial}{\partial x} \psi(x)$$

- $[\hat{x}, \hat{p}] = i\hbar$
- $\hat{T}(a) = e^{-i\hat{p}a/\hbar}$

analog: Zeitentwicklung: $U(t) = e^{-iHt/\hbar}$

Rotation um $\hat{z}$: $R(\vartheta) = e^{-iL_z\vartheta/\hbar}$

- Impulseigenzustand $|\chi_k\rangle$ zum Impuls $p = \hbar k$

besitzt Wellenfkt. $\boxed{\chi_k = e^{ikx}} = \langle \varphi_x | \chi_k \rangle$

Eigenschaften der Impulseigenzust. $\{|\chi_k\rangle\}_{k\in\mathbb{R}}$

- $\langle \chi_k | \chi_{k'} \rangle = 2\pi\, \delta(k-k')$ (Orthonormalität)
- $\mathbb{1} = \int \frac{dk}{2\pi}\, |\chi_k\rangle\langle\chi_k|$ (Vollständigkeit)
- $\hat{p} = \int \frac{dk}{2\pi}\, \hbar k\, |\chi_k\rangle\langle\chi_k|$ (Spektraldarstellung)

$\ulcorner$

- $\langle \chi_k | \chi_{k'} \rangle = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{ik'x'} = \int_{-\infty}^{\infty} dx\, e^{i(k'-k)x}$

$$= \lim_{\varepsilon\to 0} \int dx\, e^{i(k'-k)x - \frac{\varepsilon^2}{2}x^2}$$ (Gauß-Integral!)

$$= \lim_{\varepsilon\to 0} 2\pi\, \delta_\varepsilon(k-k') = 2\pi\, \delta(k-k') \quad (*)$$

- $A \overset{!}{=} \mathbb{1} \iff \langle \varphi_x | A | \varphi_{x'} \rangle \overset{!}{=} \langle \varphi_x | \varphi_{x'} \rangle$

$= \delta(x-x')$

(für alle x, x')

$\therefore \langle \varphi_x | \left(\int \frac{dk}{2\pi} |\chi_k\rangle\langle\chi_k| \right) | \varphi_{x'} \rangle$

$$= \int \frac{dk}{2\pi} \underbrace{\langle \varphi_x | \chi_k \rangle}_{= e^{ikx}} \underbrace{\langle \chi_k | \varphi_{x'} \rangle}_{= e^{-ikx'}} = \int \frac{dk}{2\pi}\, e^{ik(x-x')} \overset{(*)}{=} \delta(x-x')$$

$\lrcorner$

Impuls- und Ortsdarstellung eines Zustandes

Ortsdarstellung: $|\psi\rangle = \mathbb{1}\,|\psi\rangle$

$$= \int dx\, |\varphi_x\rangle \underbrace{\langle\varphi_x|\psi\rangle}_{\psi(x)}$$

$$\boxed{|\psi\rangle = \int dx\; \psi(x)\, |\varphi_x\rangle}$$

Impulsdarstellung: $|\psi\rangle = \mathbb{1}\,|\psi\rangle$

$$= \int \frac{dk}{2\pi}\, |\chi_k\rangle \underbrace{\langle\chi_k|\psi\rangle}_{\tilde{\psi}(k)}$$

Def.: $\boxed{\tilde{\psi}(k) := \langle\chi_k|\psi\rangle}$

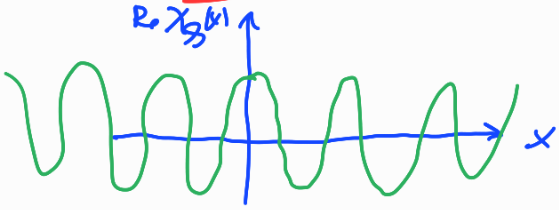
"Impulswellenfunktion" des Zustandes $|\psi\rangle$ bei Impuls $\hbar k$

Impulsdarstellung des Zust. $|\psi\rangle$

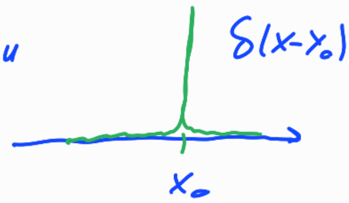
$$\boxed{|\psi\rangle = \int \frac{dk}{2\pi}\; \tilde{\psi}(k)\, |\chi_k\rangle}$$

d.h.: $\int \frac{dk}{2\pi}\, \tilde{\psi}(k)\, |\chi_k\rangle = |\psi\rangle = \int dx\, \psi(x)\, |\varphi_x\rangle$

"Welle"



"Teilchen"



"Welle-Teilchen-Dualismus"

⌜ Spin-Zust.:

$$\tilde{\alpha}\, |x+\rangle + \tilde{\beta}\, |x-\rangle = |\psi\rangle = \alpha\, |z+\rangle + \beta\, |z-\rangle$$ ⌟

Basiswechsel $\{|\varphi_x\rangle\} \Leftrightarrow \{|\chi_k\rangle\}$

$$\psi(x) \overset{?}{\Longleftrightarrow} \tilde{\psi}(k)$$

$$\tilde{\psi}(k) = \langle \chi_k | \psi \rangle = \langle \chi_k | \mathbb{1} | \psi \rangle$$

$$= \langle \chi_k | \left(\int dx\, |\varphi_x\rangle\langle\varphi_x| \right) | \psi \rangle$$

$$= \int dx\, \underbrace{\langle \chi_k | \varphi_x \rangle}_{e^{-ikx}} \underbrace{\langle \varphi_x | \psi \rangle}_{\psi(x)}$$

$\chi_k(x) = \langle \varphi_x | \chi_k \rangle$
$= e^{ikx}$

d.h.

$$\boxed{\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx \; \psi(x) \, e^{-ikx}}$$

analog:

$$\boxed{\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \tilde{\psi}(k) \, e^{+ikx}}$$

F.T.

$F.T.^{-1}$

$$\psi(x) \underset{}{\overset{F.T.}{\rightleftarrows}} \tilde{\psi}(k)$$

Skalarprodukt in Orts- und Impulsdarstellung:

Ort: $\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | \mathbb{1} | \psi_2 \rangle$

$$= \langle \psi_1 | \left(\int dx \, |\varphi_x\rangle\langle\varphi_x| \right) | \psi_2 \rangle$$

$$= \int dx \, \underbrace{\langle \psi_1 | \varphi_x \rangle}_{\psi_1(x)^*} \underbrace{\langle \varphi_x | \psi_2 \rangle}_{\psi_2(x)}$$

$$\boxed{\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} dx \; \psi_1^*(x) \, \psi_2(x)}$$

<u>Impuls:</u>

$$\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | \mathbb{1} | \psi_2 \rangle$$

$$= \langle \psi_1 | \left(\int \frac{dk}{2\pi} |\chi_k\rangle\langle\chi_k| \right) | \psi_2 \rangle$$

$$= \int \frac{dk}{2\pi} \underbrace{\langle \psi_1 | \chi_k \rangle}_{\tilde{\psi}_1^*(k)} \underbrace{\langle \chi_k | \psi_2 \rangle}_{\tilde{\psi}_2(k)}$$

$$\boxed{\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\psi}_1^*(k) \, \tilde{\psi}_2(k)}$$

insbes.:

$$|\psi|^2 = \langle \psi | \psi \rangle = \int_{-\infty}^{\infty} dx \, |\psi(x)|^2$$

$$\shortparallel$$

$$|\psi|^2 = \langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{\psi}(k)|^2$$

!

<u>Bornsche Regel:</u>

Wkt. P, das Impulsmessung einen Impuls $p \in [p_a, p_b]$ ergibt, ist

$$P = \int_{p_a/\hbar}^{p_b/\hbar} \frac{dk}{2\pi} |\underbrace{\langle \chi_k | \psi \rangle}_{\tilde{\psi}(k)}|^2$$

$p = \hbar k$

d.h. $\quad P = \int_{p_a/\hbar}^{p_b/\hbar} \frac{dk}{2\pi} |\tilde{\psi}(k)|^2 \underset{k = p/\hbar}{=} \int_{p_a}^{p_b} \frac{dp}{2\pi\hbar} |\tilde{\psi}(p/\hbar)|^2$

→ <u>Impulsaufenthaltswahrscheinlichkeitsdichte</u>: (des Zust. $|\psi\rangle$)

$$\tilde{\rho}(p) = \frac{1}{2\pi\hbar} |\tilde{\psi}(p/\hbar)|^2$$

↳ Impulserwartungswerte:

$$\langle p \rangle_{|\psi\rangle} = \int_{-\infty}^{\infty} dp \; \tilde{\rho}(p) \, p = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \, p \, |\tilde{\psi}(p/\hbar)|^2$$

$$\boxed{\langle p \rangle \underset{p = \hbar k}{=} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar k \, |\tilde{\psi}(k)|^2}$$

analog:

$$\langle p^n \rangle_\psi = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, p^n \, |\tilde{\psi}(k)|^2$$

in Ortsdarstellung:

$$\langle p \rangle_{|\psi\rangle} = \langle \psi | p | \psi \rangle$$

$$= \int dx \, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x)$$

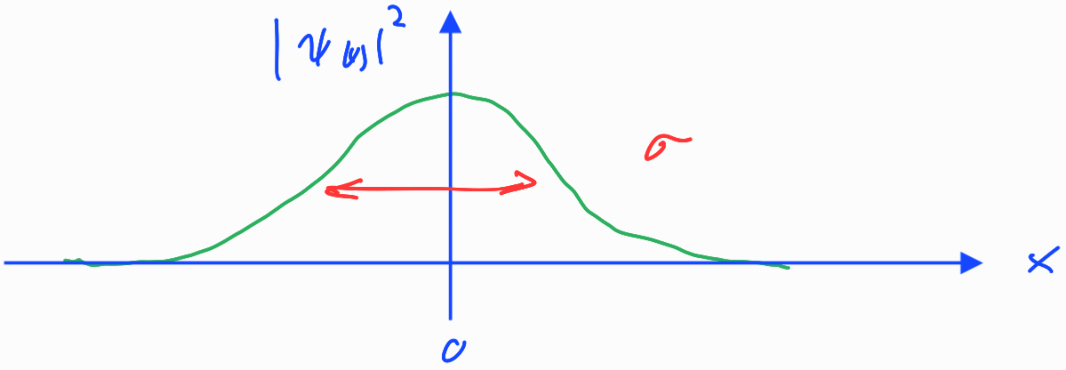
analog

$$\langle p \rangle_{|\psi\rangle} = \int dx \, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right)^n \psi(x) .$$

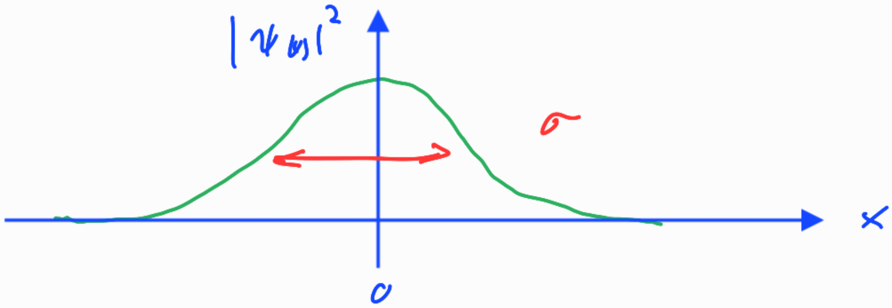
Beispiel: Zustand $|\psi\rangle$ eines Teilchens

mit Wellenfkt.

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \, e^{-x^2/4\sigma^2}$$



$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma^2}}\, e^{-x^2/4\sigma^2}$$



$$\rightarrow \quad \langle x \rangle_\psi = 0 \quad , \quad \langle x^2 \rangle_\psi = \sigma^2$$

in Impulsdarstellung: $\quad \tilde{\psi}(k) = ?$

F.T.:

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx\, \psi(x)\, e^{-ikx}$$

$$= \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt[4]{2\pi\sigma^2}}\, e^{-\frac{x^2}{4\sigma^2} - ikx}$$

$$a = \frac{1}{4\sigma^2}$$

$$= \frac{1}{\sqrt[4]{2\pi\sigma^2}} \sqrt{4\pi\sigma^2}\, e^{-\frac{k^2}{4a}}$$

$$\int dx\, e^{-ax^2 + bx} \overset{!}{=} \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

$a, b \in \mathbb{C}$, $\quad \mathrm{Re}\, a > 0$

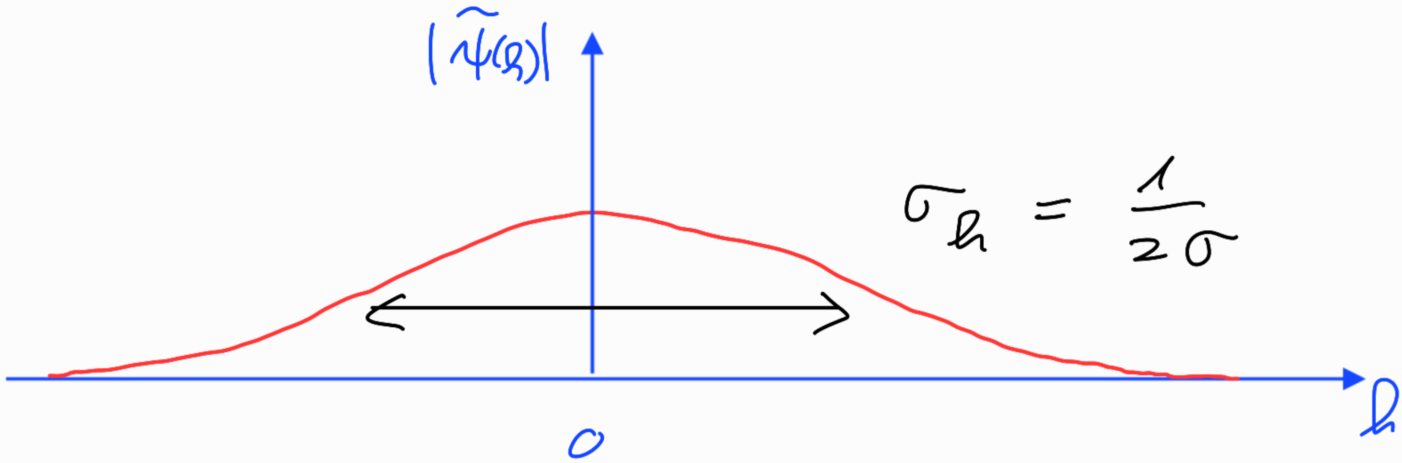
$$b = -ik$$

$$\hookrightarrow b^2 = (-i)^2 k^2 = -k^2$$

$$\rightarrow \quad \tilde{\psi}(k) = \sqrt{2}\, \sqrt[4]{2\pi\sigma^2}\, e^{-\sigma^2 k^2}$$

$$= e \quad e^{-\frac{k^2}{\underbrace{4(1/4\sigma^2)}_{\hat{=}\, \sigma_k^2}}}$$

$$\sigma_k \overset{!}{=} \frac{1}{2\sigma}$$



$$\sigma_k = \frac{1}{2\sigma}$$

$$\underbrace{\sigma}_{\Delta x}\;\underbrace{\sigma_k \hbar}_{\Delta p} = \frac{1}{2}\hbar$$

$$\Delta x\,\Delta p = \frac{\hbar}{2} \quad !$$