gestern: Stern-Gerlach-Versuch

+ elementare q.m. Beschreibung

heute: Vereinfachung + Verallgemeinerung

des Formalismus durch:

1) bequemere* Mathematik
2) Dirac-Notation

} in q.m. Anwendung

*: Lineare Algebra (komplexe VRe):

I: $\mathcal{H}$: zweidim. Zustandsraum des mag. Moments (/ Drehimpuls / Spin) eines Ag-Atoms

$\hookrightarrow$ orthogonale Zust. $\varphi_{z+}, \varphi_{z-} \in \mathcal{H}$ bilden orthonormale Basis (ONB) $B = (\varphi_{z+}, \varphi_{z-})$

Komponentendarstellung bzgl. B:

$$\varphi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightarrow \quad \varphi_{x+} = \frac{1}{\sqrt{2}}\left(\varphi_{z+} + \varphi_{z-}\right) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_B$$

$$\varphi_{x-} = \frac{1}{\sqrt{2}}\left(\varphi_{z+} - \varphi_{z-}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\varphi_{y+} = \frac{1}{\sqrt{2}}\left(\varphi_{z+} + i\varphi_{z-}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$\varphi_{y-} = \frac{1}{\sqrt{2}}\left(\varphi_{z+} - i\varphi_{z-}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

<u>Skalarprodukt in Komponenten:</u>

bisher: $\langle \varphi_{z+}, \varphi_{x-} \rangle = \langle \varphi_{z+}, \frac{1}{\sqrt{2}}(\varphi_{z+} - \varphi_{z-}) \rangle$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{\langle \varphi_{z+}, \varphi_{z+} \rangle}_{=1} - \underbrace{\langle \varphi_{z+}, \varphi_{z-} \rangle}_{=0}\Big) = 1/\sqrt{2}$$

$$\langle \varphi_{z+}, \varphi_{x-} \rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \right\rangle$$

$$= 1/\sqrt{2} + 0 \cdot (-1/\sqrt{2}) = 1/\sqrt{2}$$

$$\langle \varphi_{y-}, \varphi_{y+} \rangle = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \right\rangle$$

$$= \frac{1}{2}\left\langle \begin{pmatrix}1\\-i\end{pmatrix}, \begin{pmatrix}1\\i\end{pmatrix} \right\rangle = \frac{1}{2}\Big(\underset{1}{\underset{\shortparallel}{1^*}} \cdot 1 + \overset{i}{\overset{\shortparallel}{(-i)^*}}\, i\Big) = 0$$

- allg. Zustandsraum $\mathcal{H}$, $\dim \mathcal{H} = N$ bel. groß

<u>orthogonale</u> <u>Zust.</u> $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_N$ bilden

ONB $B = (\varphi_1, \varphi_2, \dots, \varphi_N)$

→ komponentenschreibweise:

$$\mathcal{H} \ni \psi = a_1 \varphi_1 + a_2 \varphi_2 + \dots + a_N \varphi_N = \sum_{i=1}^{N} a_i \varphi_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}_B$$

$$\mathcal{H} \ni \chi = b_1 \varphi_1 + b_2 \varphi_2 + \dots + b_N \varphi_N = \sum_{j=1}^{N} b_j \varphi_j = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}_B$$

$$\hookrightarrow \underline{\underline{\langle \psi, \chi \rangle}} = \left\langle \sum_i a_i \varphi_i , \sum_j b_j \varphi_j \right\rangle$$

$$= \sum_i \sum_j \langle a_i \varphi_i , b_j \varphi_j \rangle = \sum_i \sum_j a_i^* b_j \underbrace{\langle \varphi_i , \varphi_j \rangle}_{= \delta_{ij} = \begin{cases} 0 : i \neq j \\ 1 : i = j \end{cases}}$$

$$= \underline{\underline{\sum_i a_i^* b_i}} = \underline{\underline{\left\langle \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \right\rangle}}$$

- <u>Dualvektor</u>

im Komponenten bzgl. ONB $\mathcal{B}$:

Vektor $\psi = \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}}_{N\times 1 \text{ Matrix}} \longrightarrow$ **Dualvektor**

$$\psi^+ = \underbrace{(a_1^*, a_2^*, \dots, a_N^*)}_{1\times N \text{ Matrix}}$$

$$\psi^+ \chi = (a_1^*, a_2^*, \dots, a_N^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots + a_N^* b_N$$

$$= \sum_{i=1}^{N} a_i^* b_i = \langle \psi, \chi \rangle$$

Def. <u>ohne</u> ONB:

Vektor $\psi \longrightarrow$ Dualvektor ψ^+

$$:= \text{Abb}: \mathcal{H} \to \mathbb{C}$$
$$\chi \mapsto \langle \psi, \chi \rangle$$

III Dirac-Notation:

bisherige Notation		Dirac-Notation
Vektor ψ	$\longleftrightarrow$	$\|\psi\rangle$: "ket ψ"
Dualvektor ψ^+	$\longleftrightarrow$	$\langle\psi\|$: "Bra ψ"
$\psi^+\chi$	$\longleftrightarrow$	$\langle\psi\|\|\chi\rangle \equiv \langle\psi\|\chi\rangle \equiv \langle\psi,\chi\rangle$

"Bra(c)ket"

Beispiele:

- $|\varphi_{z+}\rangle$, $|\varphi_{z-}\rangle$ Zustände des mag. Moments

$$\left.\begin{aligned}
|\varphi_{x+}\rangle &= \frac{1}{\sqrt{2}}\left(|\varphi_{z+}\rangle + |\varphi_{z-}\rangle\right)\\
|\varphi_{x-}\rangle &= \frac{1}{\sqrt{2}}\left(|\varphi_{z+}\rangle - |\varphi_{z-}\rangle\right)\\
|\varphi_{y+}\rangle &= \frac{1}{\sqrt{2}}\left(|\varphi_{z+}\rangle + \underline{i}\,|\varphi_{z-}\rangle\right)\\
|\varphi_{y-}\rangle &= \frac{1}{\sqrt{2}}\left(|\varphi_{z+}\rangle \underline{-i}\,|\varphi_{z-}\rangle\right)
\end{aligned}\right\}\text{"kets"}$$

$\hookrightarrow$

$$\langle\varphi_{x+}| = \frac{1}{\sqrt{2}}\left(\langle\varphi_{z+}| + \langle\varphi_{z-}|\right)$$

$$\vdots$$

$$\langle\varphi_{y+}| = \frac{1}{\sqrt{2}}\left(\langle\varphi_{z+}| \underline{-i}\,\langle\varphi_{z-}|\right)$$

$$\langle\varphi_{y-}| = \frac{1}{\sqrt{2}}\left(\langle\varphi_{z+}| + i\,\langle\varphi_{z-}|\right)$$

$$\langle \varphi_{z+} | \varphi_{y-} \rangle = \langle \varphi_{z+} | \left(\frac{1}{\sqrt{2}} \left(|\varphi_{z+}\rangle + i\, |\varphi_{z-}\rangle \right) \right)$$

$$= \frac{1}{\sqrt{2}} \underbrace{\langle \varphi_{z+} | \varphi_{z+} \rangle}_{=1} + \frac{1}{\sqrt{2}}\, i \underbrace{\langle \varphi_{z+} | \varphi_{z-} \rangle}_{=0} = 1/\sqrt{2} \, .$$

Bemerkungen:

- $\langle \varphi | \psi \rangle = \langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle^* = \langle \psi | \varphi \rangle^*$

$$\boxed{\langle \varphi | \psi \rangle = \langle \psi | \varphi \rangle^*}$$

- oft: $$|\langle \varphi | \psi \rangle|^2 = \langle \varphi | \psi \rangle \langle \varphi | \psi \rangle^* = \langle \varphi | \underline{\underline{\psi \rangle \langle \psi}} | \varphi \rangle$$

IV Lineare Abbn, Operatoren, Adjunktion, hermitsche Operatoren

Lineare Abb. $:\overset{!}{=}$ Abb. $A : \mathcal{H}_1 \to \mathcal{H}_2$, $\varphi \mapsto A(\varphi)$

$(\mathcal{H}_1, \mathcal{H}_2$ VRe$)$

mit Eigenschaften

(i) $A(\varphi + \psi) \overset{!}{=} A(\varphi) + A(\psi)$

(ii) $A(\lambda\varphi) = \lambda A(\varphi)$

Notation: bei lin. Abb keine Argumentklammern:

$$\underline{A\varphi} \quad (\text{statt } A(\varphi))$$

Für lin. Abb. $A, B : \mathcal{H}_1 \to \mathcal{H}_2$ gilt es

Addition: $(A+B)\varphi := A\varphi + B\varphi$

Skalarmultiplikation: $(\underset{\in \mathbb{C}}{\lambda} A)\varphi := \lambda A\varphi$

- Adjunktion einer lin. Abb $A : \mathcal{H}_1 \to \mathcal{H}_2$

Analogie: komplexe Konjugation:

$$\langle \varphi, \lambda\psi \rangle \overset{!}{=} \langle \lambda^* \varphi, \psi \rangle$$

$$\lambda \longrightarrow \lambda^*$$

lin. Abb. $A: \mathcal{H}_1 \to \mathcal{H}_2$

$\Rightarrow$ adjungierte lin Abb. $A^+: \mathcal{H}_2 \to \mathcal{H}_1$

def. durch:

$$\boxed{\langle \varphi, A\psi \rangle = \langle A^+\varphi, \psi \rangle}$$

(für alle $\varphi \in \mathcal{H}_2$, $\psi \in \mathcal{H}_1$)

- $\boxed{\text{lin. Abb } A: \mathcal{H} \to \mathcal{H} \text{ hermitesch } :\Leftrightarrow A = A^+}$

- lin. Abb. $A: \mathcal{H} \to \mathcal{H}$ $:\Leftrightarrow$ A Operator auf $\mathcal{H}$

<u>Beispiele:</u>

$$\left.\begin{array}{l} |\psi\rangle, |\varphi\rangle \in \mathcal{H} \\ \langle\psi|, \langle\varphi| \end{array}\right\} \to \text{Operatoren auf } \mathcal{H}:$$

z.B.: $A := |\varphi\rangle\langle\psi|$, $B := |\psi\rangle\langle\varphi|$

Beispiele:

$$\left.\begin{array}{l} |\psi\rangle,\ |\varphi\rangle \in \mathcal{H} \\ \langle\psi|,\ \langle\varphi| \end{array}\right\} \longrightarrow \text{Operatoren auf } \mathcal{H}:$$

z.B.: $A := |\varphi\rangle\langle\psi|$, $B := \underline{\underline{|\psi\rangle\langle\varphi|}}$

$$A|\chi\rangle = \overbrace{|\varphi\rangle\langle\psi|}^{A}\chi\rangle = \langle\psi|\chi\rangle\, |\varphi\rangle$$

$$B|\chi\rangle = \overbrace{|\psi\rangle\langle\varphi|}^{B}\chi\rangle = \langle\varphi|\chi\rangle\, |\psi\rangle$$

$$(A+B)|\chi\rangle = A|\chi\rangle + B|\chi\rangle = \langle\psi|\chi\rangle|\varphi\rangle + \langle\varphi|\chi\rangle|\psi\rangle$$

$$(\lambda B)|\chi\rangle = \lambda B|\chi\rangle = \lambda\langle\varphi|\chi\rangle\,|\psi\rangle$$