gestern:

3. Postulat: Dynamik

$$\boxed{\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H |\psi(t)\rangle} \qquad \boxed{i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle}$$

Schrödingergleichung

- H : Hamilton-Operator $\equiv$ Energie-Op. $(H = H^+)$
- $\hbar = 1{,}054 \ldots \cdot 10^{-34}$ Js

$$\rightsquigarrow \quad \boxed{\frac{d}{dt} \langle A \rangle_{\psi(t)} = \left\langle \frac{i}{\hbar} [H, A] \right\rangle_{\psi(t)}}$$

Kommutator: $[A, B] := AB - BA$

$$\rightsquigarrow \quad \boxed{A \text{ Erhaltungsgröße} \iff [H, A] = 0}$$

Lösung der Schrödingergleichung zum Anfangszustand $|\psi(0)\rangle = |\psi_0\rangle$:

$$\boxed{|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|\psi_0\rangle}$$

[analog $\dot{y} = -i\omega y$

$\rightarrow \quad y(t) = e^{-i\omega t}\, y_0$]

explizit in "Energiedarstellung":

d.h.:

- $$H = \sum_{n=0}^{N} E_n\, |\varphi_n\rangle\langle\varphi_n|$$

Eigenenergien $E_0 \le E_1 \le E_2 \ldots$ (Eigenwerte)

Energieeigenzustände $|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle \ldots$ (Eigenvektoren) von H

$$\boxed{H\,|\varphi_n\rangle = E_n\,|\varphi_n\rangle}$$

"stationäre Schrödingergl." (Eigenwertgleichung)

$$\rightarrow \boxed{e^{-\frac{i}{\hbar}Ht} \equiv \exp\left(-\frac{i}{\hbar}Ht\right) = \sum_{n=0}^{N} e^{-i\omega_n t}\,|\varphi_n\rangle\langle\varphi_n|}$$

$\omega_n = E_n/\hbar$

$\rightarrow$

$$\longrightarrow \boxed{e^{-\frac{i}{\hbar}Ht} \equiv \exp\left(-\frac{i}{\hbar}Ht\right) = \sum_{n=0}^{N} e^{-i\omega_n t} |\varphi_n\rangle\langle\varphi_n|} \quad (*)$$

$$\omega_n = E_n/\hbar$$

- $$|\psi_0\rangle = |\varphi_k\rangle \xrightarrow{t} |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\varphi_k\rangle$$

$$= e^{-i\omega_k t}|\varphi_k\rangle$$

Energieeigen-zustand

$$e^{-\frac{i}{\hbar}Ht}|\varphi_k\rangle = \overbrace{\sum_{n=0}^{N} e^{-i\omega_n t}|\varphi_n\rangle\langle\varphi_n|}^{e^{-iHt/\hbar} \;\overset{*}{=}}\;\underbrace{\langle\varphi_n|\varphi_k\rangle}_{\hat{=}\,\delta_{nk}} = e^{-i\omega_k t}|\varphi_k\rangle$$

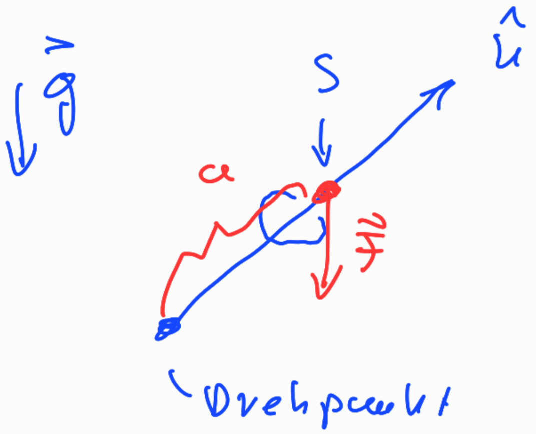
- $$|\psi_0\rangle = \sum_k a_k|\varphi_k\rangle \xrightarrow{t} |\psi(t)\rangle = \sum_k a_k e^{-\frac{i}{\hbar}Ht}|\varphi_k\rangle$$

$$= \sum_k a_k e^{-i\omega_k t}|\varphi_k\rangle$$

Dynamik eines Spins im Magnetfeld

$\Rightarrow$ mag. Moment $\vec{\mu}$

klassisches Analogon: Präzession eines "schnellen" symmetrischen Kreisels:



$$\underline{\vec{L}} = L\hat{u} \quad ✓$$

Drehmoment: $\vec{M} = a\hat{u} \times \vec{F}$

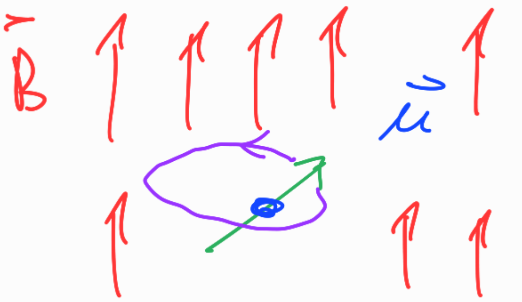
$$= \frac{a}{L}\vec{L} \times \vec{F}$$

$$\vec{F} = m\vec{g}$$

$$\vec{M} = \frac{am}{L}\underline{\vec{L}} \times \vec{g}$$

$$\underline{\dot{\vec{L}}} = \vec{M} = \frac{am}{L}\underline{\vec{L} \times \vec{g}}$$

mag. Moment im B-Feld:

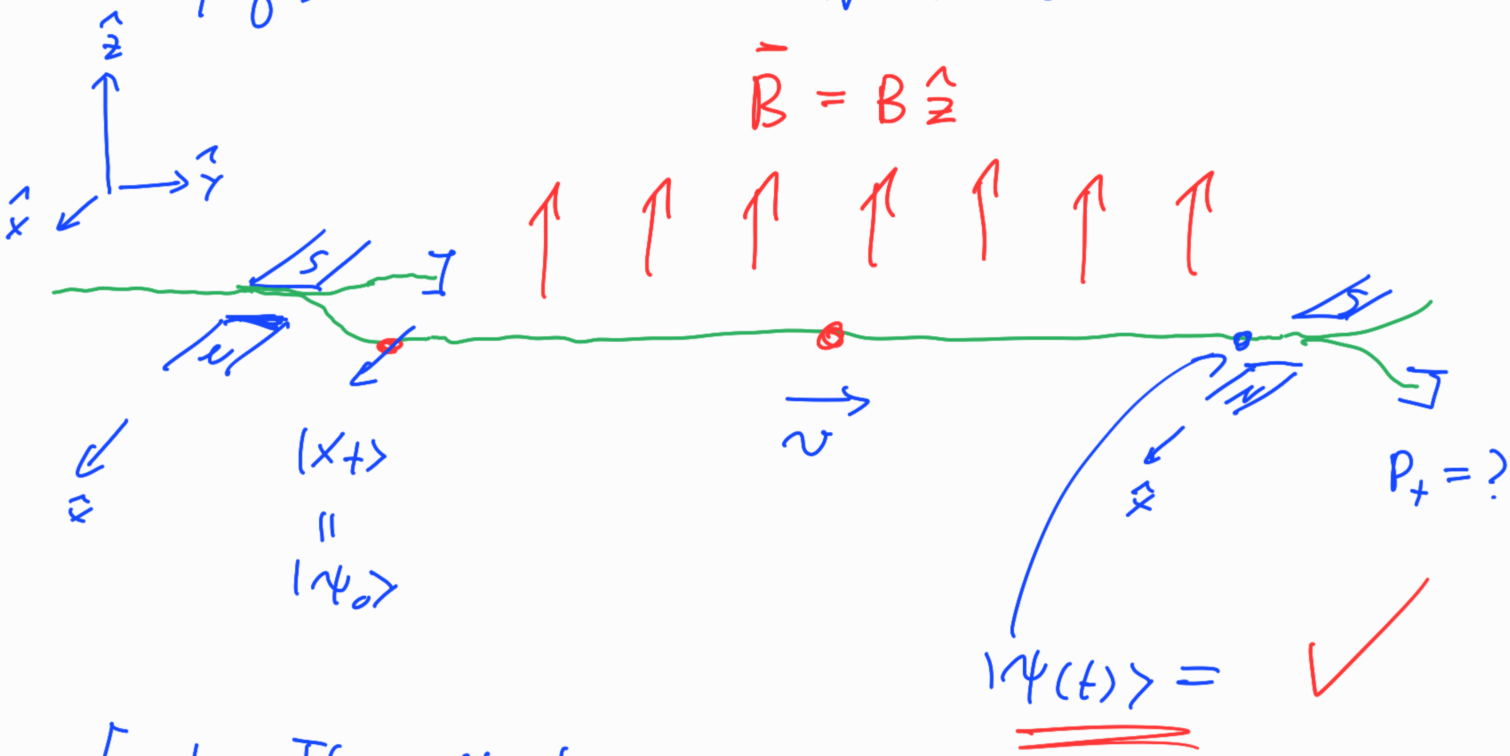


$$\vec{M} = \vec{B} \times \vec{\mu}$$

$$\underline{\dot{\vec{\mu}}(t)} \sim \dot{\vec{L}} = \vec{M} = \underline{\vec{B} \times \vec{\mu}(t)}$$

Larmorpräzession

2-stufiges Stern-Gerlach-Experiment:



[$t =$ Flugzeit]

Hamilton-Funktion: $\qquad H \overset{!}{=} -\vec B \cdot \vec \mu$

$$H = -B\,\mu_z$$

$$(\vec B = B\,\hat z)$$

Hamilton-Operator: $\qquad \hat H = -B\,\hat\mu_z$

$$\hat\mu_z = \frac{\mu_0}{2}|z+\rangle\langle z+| - \frac{\mu_0}{2}|z-\rangle\langle z-| \leftarrow \begin{cases} +\mu_0/2, & -\mu_0/2 \\ |z+\rangle & |z-\rangle \end{cases}$$

$$\rightarrow \boxed{H = -\frac{B\mu_0}{2}|z+\rangle\langle z+| + \frac{B\mu_0}{2}|z-\rangle\langle z-|}$$

$$|\psi_0\rangle = |x+\rangle \xrightarrow{t} |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|x+\rangle$$

$$\rightarrow \boxed{H = -\frac{B\mu_0}{2}|z+\rangle\langle z+| + \frac{B\mu_0}{2}|z-\rangle\langle z-|}$$

$$|\psi_0\rangle = |x+\rangle \xrightarrow{t} |\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|x+\rangle$$

$$e^{-\frac{iHt}{\hbar}} = \exp\left(\frac{iB\mu_0}{2\hbar} t\, |z+\rangle\langle z+| - i\frac{B\mu_0}{2\hbar} t\, |z-\rangle\langle z-| \right)$$

$$= \exp\left(i \frac{\omega}{2} t\, |z+\rangle\langle z+| - i\frac{\omega}{2} t\, |z-\rangle\langle z-| \right)$$

mit $\boxed{\omega = \frac{B\mu_0}{\hbar}}$ (Larmorfrequenz)

$$\rightarrow e^{-\frac{iHt}{\hbar}} = e^{i\frac{\omega}{2}t}|z+\rangle\langle z+| + e^{-i\frac{\omega}{2}t}|z-\rangle\langle z-|$$

$$\rightarrow |\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|x+\rangle$$

$|x+\rangle = (|z+\rangle + |z-\rangle)/\sqrt{2}$: $|z+\rangle = (|x+\rangle + |x-\rangle)/\sqrt{2}$

$|x-\rangle = (|z+\rangle - |z-\rangle)/\sqrt{2}$: $|z-\rangle = (|x+\rangle - |x-\rangle)/\sqrt{2}$

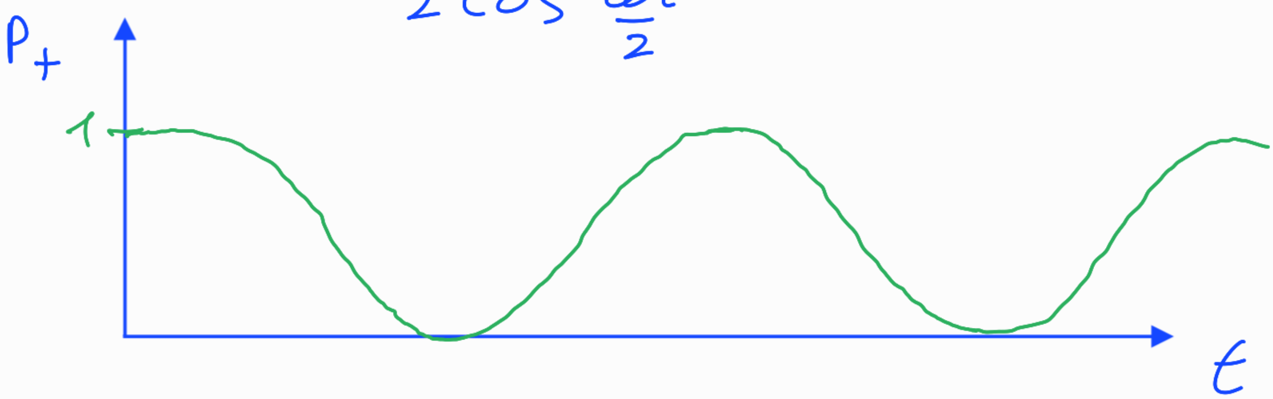
$$|\psi(t)\rangle = e^{i\frac{\omega t}{2}}|z+\rangle\underbrace{\langle z+|x+\rangle}_{1/\sqrt{2}} + e^{-i\frac{\omega t}{2}}|z-\rangle\underbrace{\langle z-|x+\rangle}_{1/\sqrt{2}}$$

$$|\psi(t)\rangle = e^{i\frac{\omega t}{2}} |z+\rangle \underbrace{\langle z+|x+\rangle}_{1/\sqrt{2}} + e^{-i\frac{\omega t}{2}} |z-\rangle \underbrace{\langle z-|x+\rangle}_{1/\sqrt{2}}$$

d.h. $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{i\frac{\omega t}{2}} |z+\rangle + e^{-i\frac{\omega t}{2}} |z-\rangle \right)$$

$$\rightarrow \quad P_+ = |\langle x+|\psi(t)\rangle|^2 = \frac{1}{2}\left| \frac{1}{\sqrt{2}} e^{i\frac{\omega t}{2}} + \frac{1}{\sqrt{2}} e^{-i\frac{\omega t}{2}} \right|^2$$

$$= \frac{1}{4}\Big| \underbrace{e^{i\frac{\omega t}{2}} + e^{-i\frac{\omega t}{2}}}_{2\cos\frac{\omega t}{2}} \Big|^2 = \cos^2\frac{\omega t}{2} \quad !$$



$$\langle \mu_x \rangle_t = \langle \psi(t) | \hat{\mu}_x | \psi(t) \rangle$$

$$= P_+ \cdot \left(+\frac{\mu_0}{2}\right) + \overset{1-P_+}{\overset{\parallel}{P_-}} \cdot \left(-\frac{\mu_0}{2}\right)$$

$$= P_+ \cdot \mu_0 - \frac{\mu_0}{2} = \frac{\mu_0}{2}\Big(\underbrace{2\cos^2\left(\frac{\omega t}{2}\right) - 1}_{\overset{(*)}{=} \cos\omega t}\Big)$$

Γ NR: $\quad 2\underline{\cos^2\left(\frac{\alpha}{2}\right)} - 1$ $\quad (*)$

$$= \cos^2\left(\tfrac{\alpha}{2}\right) + \cos^2\left(\tfrac{\alpha}{2}\right) - 1$$

$$= \cos^2\left(\tfrac{\alpha}{2}\right) + \cancel{1} - \sin^2\left(\tfrac{\alpha}{2}\right) - \cancel{1}$$

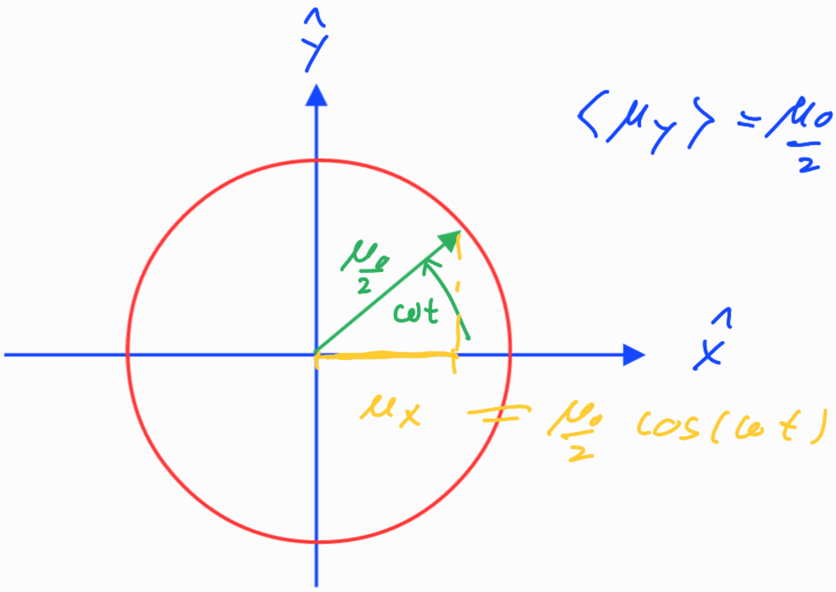
$$= \cos^2\tfrac{\alpha}{2} - \sin^2 \alpha/2$$

$$= \underline{\cos(\alpha)}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$\lrcorner$

$$\boxed{\langle \mu_x \rangle_t = \frac{\mu_0}{2} \cos \underline{\omega t}}$$



$$\langle \mu_y \rangle = \frac{\mu_0}{2} \sin \omega t$$

$$\langle \mu_z \rangle_t \overset{?}{=} \langle \mu_z \rangle_0$$

$$\hat{H} = -B\hat{\mu}_z \longrightarrow [H, \mu_z]$$

$$= [B\mu_z, \mu_z]$$

$$= -B[\mu_z, \mu_z] = 0 \;!$$

$$\vec{B} \parallel \hat{z} \longrightarrow \hat{\mu}_z$$ Erhaltungsgröße!