Lösungshinweise Blatt 1

2a) • $\mathrm{Re}\,(1+2i) = 1$, $\mathrm{Im}\,(1+2i) = 2$, $|1+2i| = \sqrt{5}$;

• $\frac{1}{1+i} = \frac{1-i}{2}$, $\rightarrow$ $\mathrm{Re}\left(\frac{1}{1+i}\right) = \frac{1}{2}$, $\mathrm{Im}\left(\frac{1}{1+i}\right) = -\frac{1}{2}$, $\left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}}$

• $\frac{1}{i} = -i$ $\rightarrow$ $\mathrm{Re}\,1/i = 0$, $\mathrm{Im}\,1/i = -i$, $|1/i| = 1$

• $z = (1+2i)(1-3i) = 7-i$ $\rightarrow$ $\mathrm{Re}\,z = 7$, $\mathrm{Im}\,z = -1$

$|z| = \sqrt{50}$,

• $e^{i\pi/4} = \cos\pi/4 + i\sin\pi/4 = \frac{1+i}{\sqrt{2}}$ $\rightarrow$ $\mathrm{Re}\,e^{i\pi/4} = 1/\sqrt{2}$,

$\mathrm{Im}\,e^{i\pi/4} = 1/\sqrt{2}$,

$|e^{i\pi/4}| = 1$;

• $\sqrt{-9} = \sqrt{9\cdot(-1)} = 3i$ $\rightarrow$ $\mathrm{Re}\sqrt{-9} = 0$, $\mathrm{Im}\sqrt{-9} = 3$

$|\sqrt{-9}| = 3$

b)

$$e^{i\varphi} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\varphi)^n = \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!}(i\varphi)^{2\ell} + \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!}(i\varphi)^{2\ell+1}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!}\varphi^{2\ell} + i\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)}\varphi^{2\ell+1} = \cos\varphi + i\sin\varphi .$$

c)

$$e^{ix} + e^{-ix} = \cos x + i\sin x + \cos(-x) + i\sin(-x) = 2\cos x\,,$$

$$e^{ix} - e^{-ix} = \cos x + i\sin x - \left(\cos(-x) + i\sin(-x)\right) = 2i\sin x\,.$$

d)

$$\underset{x\in\mathbb{R}}{1} = |e^{ix}|^2 = |\cos x + i\sin x|^2 = \cos^2 x + \sin^2 x$$

$$(\cos x)' = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)' = \frac{i}{2}\left(e^{ix} - e^{-ix}\right) = -\sin x$$

$$(\sin x)' = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)' = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x\,.$$

3a)

$$\langle \varphi_1, \varphi_1 \rangle = 1\,, \quad \langle \varphi_1, \varphi_2 \rangle = 0\,, \quad \langle \varphi_1, i\varphi_2 \rangle = i$$

$$\langle i\varphi_1, \varphi_2 \rangle = -i\,,$$

$$\langle a_1\varphi_1 + a_2\varphi_2, b_1\varphi_1 + b_2\varphi_2 \rangle = a_1^* b_1 + a_2^* b_2$$

b)

$$\langle \psi_1, \psi_1 \rangle = \frac{1}{2} + \frac{1}{2} = 1\,, \quad \langle \psi_2, \psi_2 \rangle = \frac{1}{2} + \frac{1}{2} = 1\,,$$

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{2}\langle \varphi_1 + i\varphi_2, \varphi_1 - i\varphi_2 \rangle = \frac{1}{2}\Big(\underbrace{\langle \varphi_1, \varphi_1 \rangle}_{=1} + \underbrace{\langle i\varphi_2, -i\varphi_2 \rangle}_{=(-i)(-i)=-1}\Big) = 0\,.$$

4 a)

$$\|\chi_1\|^2 = \frac{1}{2}\langle \varphi_1 + i\varphi_2, \varphi_1 + i\varphi_2\rangle = \frac{1}{2}(1+1) = 1$$

$$\|\chi_2\|^2 = \frac{1}{2}\langle \varphi_1 - i\varphi_2, \varphi_1 - i\varphi_2\rangle = \frac{1}{2}(1+1) = 1$$

$$\|\chi_3\|^2 = \frac{1}{5}\langle 2\varphi_1 + \varphi_2, 2\varphi_1 + \varphi_2\rangle = \frac{1}{5}(4+1) = 1$$

b)

$$P_i = |\langle \varphi_1, \chi_1\rangle|^2 = \left|\left\langle \varphi_1, \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}\right\rangle\right|^2 = 1/2$$

$$P_{ii} = |\langle \chi_1, \varphi_1\rangle|^2 = |\langle \varphi_1, \chi_1\rangle|^2 = P_i = 1/2$$

$$P_{iii} = |\langle \chi_1, \chi_2\rangle|^2 = 0$$

$$P_{iv} = |\langle \chi_1, \chi_3\rangle|^2 = \frac{1}{10}|\langle \varphi_1 + i\varphi_2, 2\varphi_1 + \varphi_2\rangle|^2$$

$$= \frac{1}{10}|2 - i|^2 = 1/2$$

$$P_v = |\langle \varphi_1, \chi_3\rangle|^2 = \frac{1}{5}|\langle \varphi_1, 2\varphi_1 + \varphi_2\rangle|^2 = 4/5 .$$