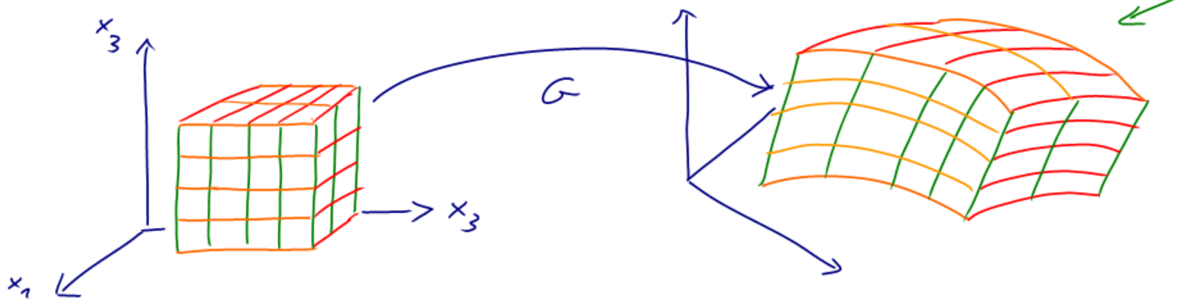
Mehrdimensionale Integration: Jacobi-Determinante, Transformationssatz

Erinnerung: 3-dim. Integral über ein Volumgebiet $G \subset \mathbb{R}^3$



Parametrisierung: $G : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \longrightarrow \mathbb{R}^3$

$$x = (x_1, x_2, x_3) \longmapsto G(x)$$

$$\hookrightarrow \quad \int_G f \, dV := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(G(x)) \underbrace{\left| \Omega_3\left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_3} \right) \right| dx_1 \, dx_2 \, dx_3}_{\widehat{=}\ dV}$$

n-dimensionales Volumengebiet $G \subset \mathbb{R}^n$ parametrisiert durch

$$G : [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \longrightarrow \mathbb{R}^n$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_n(x) \end{pmatrix}$$

n-dimensionales Integral von $f : \mathbb{R}^n \to \mathbb{R}$ über G:

$$\int_G f \, dV := \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(G(x)) \underbrace{\left| \Omega_n \left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_n} \right) \right| dx_1 \, dx_2 \cdots dx_n}$$

$= dV$: n-dim. Volumenelement

- $dx_1\, dx_2 \cdots dx_n \equiv d^n x$

- <u>Differenzial / Ableitung / Jacobi-Matrix</u> (vgl. Analysis I/II) :

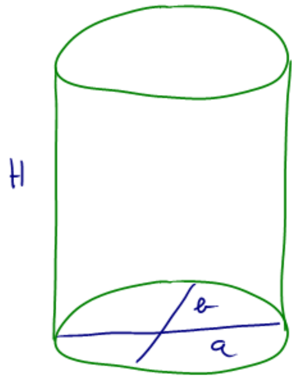
$$dG : \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$$

$$x \longmapsto dG_x = \left(\frac{\partial G}{\partial x_1}, \dots, \frac{\partial G}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \frac{\partial G_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial G_n}{\partial x_1} & & \cdots & \frac{\partial G_n}{\partial x_n} \end{pmatrix}$$

- $\Omega_n \left(\frac{\partial G}{\partial x_1}, \dots, \frac{\partial G}{\partial x_n} \right) = \det \left(\frac{\partial G}{\partial x_1}, \dots, \frac{\partial G}{\partial x_n} \right) = \det (dG)$ ← <u>Jacobi-Determinante</u>

$$\boxed{\int_G f \, dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(G(x)) \, \left| \det (dG) \right| \, d^n x}$$

Beispiel: Zylinder über Ellipse mit Halbachsen a, b



$$G : [0,1]\times[0,2\pi]\times[0,H] \longrightarrow \mathbb{R}^3$$

$$(u,\varphi,z) \longmapsto G(u,\varphi,z) = \begin{pmatrix} a\,u\cos\varphi \\ b\,u\sin\varphi \\ z \end{pmatrix}$$

$$\rightsquigarrow \det(dG) = \det\left(\frac{\partial G}{\partial u}, \frac{\partial G}{\partial \varphi}, \frac{\partial G}{\partial z}\right)$$

$$= \det\begin{pmatrix} a\cos\varphi & -au\sin\varphi & 0 \\ b\sin\varphi & bu\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = abu$$

$$\rightsquigarrow \operatorname{Vol}(G) = \int_G 1\,dV = \int_0^1\int_0^{2\pi}\int_0^H \det(dG)\,du\,d\varphi\,dz = \int_0^1\int_0^{2\pi}\int_0^H ab\,u\,du\,d\varphi\,dz$$

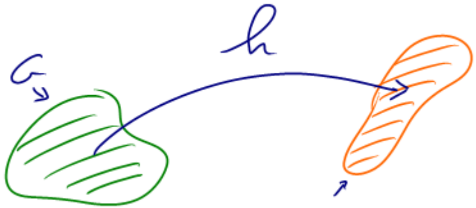
$$= ab\int_0^1 u\,du\int_0^{2\pi} d\varphi\int_0^H dz = \pi ab\cdot H$$

Transformationssatz

$\equiv$ mehrdimensionale Version der Substitutionsregel

$$\int_{h(a)}^{h(b)} f(x)\, dx = \int_a^b f \circ h(x)\, h'(x)\, dx \quad :$$

- $G \subset \mathbb{R}^n$ Volumengebiet
- $h : G \to \mathbb{R}^n$ injektiv, diffbar
- $f : \mathbb{R}^n \to \mathbb{R}^n$



$h(G) \equiv \{ h(x) \mid x \in G \}$

$$\boxed{\int_{h(G)} f\, dV \overset{!}{=} \int_G f \circ h \, |\det(dh)|\, dV}$$

Transformationssatz folgt im wesentlichen aus allg. Kettenregel und Matrixmultiplikationssatz für Determinanten:

$G \subset \mathbb{R}^n$ sei parametrisiert durch $G : [a_1, b_1] \times \cdots \times [a_n, b_n] \longrightarrow \mathbb{R}^n$;

dann $h \circ G : [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}^n$ offenbar Parametrisierung von $h(G)$ und somit

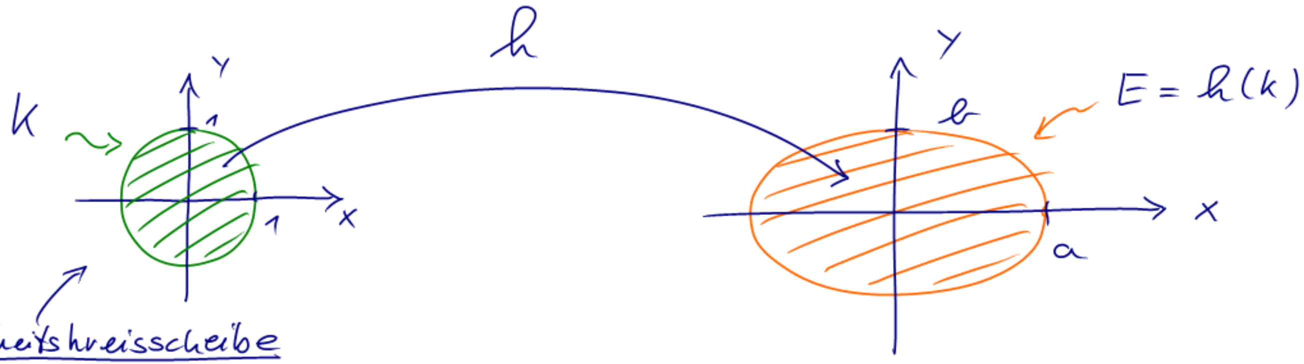
$$\int_{h(G)} f \, dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(h \circ G(x)) \, |\det(d(h \circ G))| \, d^n x$$

nach Kettenregel ist $d(h \circ G)_x = dh_{G(x)} \cdot dG_x$

d.h. $\det(d(h \circ G)_x) \overset{!}{=} \det(dh_{G(x)}) \cdot \det(dG_x)$,

$$\Rightarrow \int_{h(G)} f \, dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} (f \circ h)(G(x)) \, |\det(dh_{G(x)}) \det(dG_x) \, d^n x| = \int_G f \circ h \, |\det(dh)| \, dV$$

Anwendungsbeispiele: 1) Flächeninhalt einer Ellipse



Einheitskreisscheibe

$$x^2 + y^2 \le 1$$

Ellipse E, Halbachsen a, b:

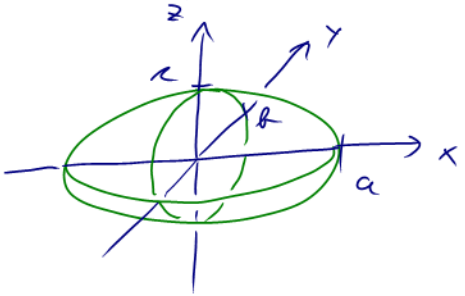
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$

Trafo $K \to E$ mittels $\boxed{h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}}$,

$\rightarrow \quad dh = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \det(dh) = ab$

$$\rightarrow \quad \underline{\underline{Fl(E)}} = \int_E df = \int_{h(K)} df \overset{T.S.}{=} \int_K |\det(dh)|\, df = ab \underbrace{\int_K df}_{= \pi} = \underline{\underline{\pi a b}}$$

2) Hauptträgheitsmomente eines Ellipsoids mit Halbachsen a, b, c:



$$E = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

E ist Bild der Einheitskugel $K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x^2+y^2+z^2 \leq 1 \right\}$

unter Abb. $h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$: $E = h(K)$.

$\Rightarrow \det(dh) = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$,

$$\rightarrow I_x = \rho_0 \int_{E=h(K)} (y^2+z^2)\, dV \overset{\text{T.S.}}{=} \rho_0 \int_K (b^2y^2+c^2z^2)\, abc\, dV \longrightarrow$$

d.h. $\quad I_x = \rho_0 \, abc \left(b^2 \int_K y^2 \, dV + c^2 \int_K z^2 \, dV \right)$

$$\int_K y^2 \, dV = \int_K z^2 \, dV \underset{\text{Kugelkoordinaten}}{=} \int_0^1 \int_0^{\pi} \int_0^{2\pi} (r \cos\vartheta)^2 \, r^2 \sin\vartheta \, d\varphi \, d\vartheta \, dr$$

$$= \int_0^1 r^4 \, dr \int_0^{2\pi} d\varphi \int_0^{\pi} \underbrace{\sin\vartheta \cos^2\vartheta}_{= -\frac{1}{3}\frac{d}{d\vartheta}\cos^3\vartheta} \, d\vartheta = \frac{1}{5} \cdot 2\pi \cdot \frac{2}{3} = \frac{4\pi}{3} \cdot \frac{1}{5}$$

$$\rightarrow \quad I_x = \rho_0 \underbrace{\frac{4\pi}{3} abc}_{= \mathrm{Vol}(E)} \cdot \frac{1}{5} (b^2 + c^2) = \frac{1}{5} (b^2 + c^2) M$$

analog: $\quad I_y = \frac{1}{5} (a^2 + c^2) M \quad , \quad I_z = \frac{1}{5} (a^2 + b^2) M$