Anwendungsbeispiele:

1) Berechnung von $I(k,a) := \int\limits_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx$, $\quad k \in \mathbb{R}$, $a \in \mathbb{R}_+$

Fasse $I(k,a)$ als kompl. Kurvenintegral über reelle Achse $\mathbb{R} \subset \mathbb{C}$ auf:

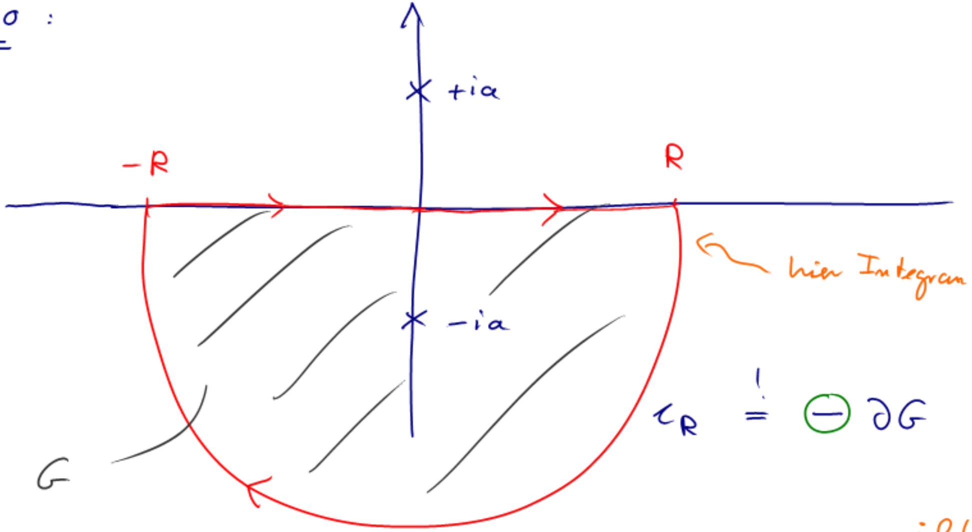
$$I(k,a) = \int\limits_{\mathbb{R}} \underbrace{\frac{e^{-ikz}}{(z+ia)(z-ia)}}_{\text{meromorph, Pole 1. Ordnung bei } \mp ia} dz$$

Idee: finde geschlossenen Weg $\mathcal{C}_R \subset \mathbb{C}$ so, dass

$$I(k,a) \stackrel{!}{=} \lim_{R\to\infty} \underbrace{\int\limits_{\mathcal{C}_R} \frac{e^{-ikz}}{(z+ia)(z-ia)} dz}_{\text{mittels Residuensatz bestimmbar}}$$

!

sei $k>0$:



hier Integrand $\sim \frac{1}{R^2}$ ✓

$c_R \overset{!}{=} \ominus\, \partial G$

hier Integrand $\sim e^{-ik(-iR)} = e^{-kR}$

exponentiell klein ✓

$$\longrightarrow \quad I(k,a) \overset{!}{=} \lim_{R\to\infty} \int_{c_R} \frac{e^{-ikz}}{(z-ia)(z+ia)}\,dz = \ominus\, 2\pi i \;\mathrm{Res}(\dots, -ia)$$

$$= -2\pi i \, \frac{e^{-ka}}{-2ia} = \frac{\pi}{a}\, e^{-ka}$$

Falls $k < 0$ schließt man Weg im oberen Halbebene

$$\Rightarrow \quad I(k, a) = \frac{\pi}{a} e^{+ka} \, ;$$

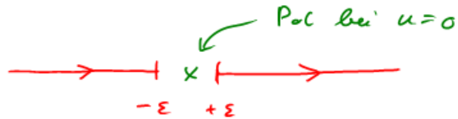
d.h.

$$\boxed{I(k, a) \equiv \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2 + a^2} \, dx = \frac{\pi}{a} e^{-|k| a}} \, .$$

2) $$J(b) := \int_{-\infty}^{\infty} du \; \frac{e^{ibu}}{u} \equiv \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} du \; \frac{e^{ibu}}{u} + \int_{+\varepsilon}^{+\infty} du \; \frac{e^{ibu}}{u} \right)$$

„Hauptwert" des Integrals $\qquad \underline{(b > 0)}$

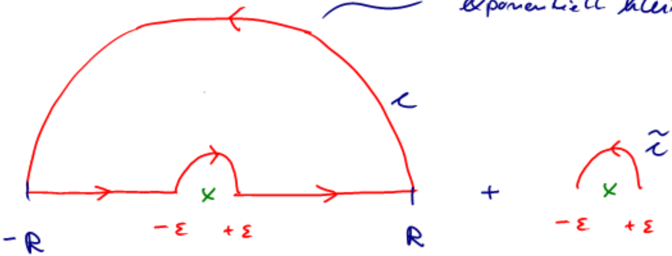
$\hat{=}$



exponentiell kleiner Integrand

$(b>0)$

$=$

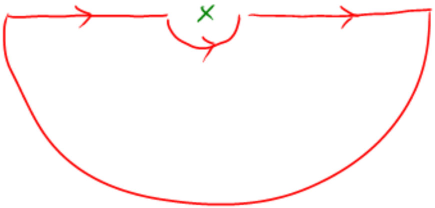


$\tilde{c} : [0,\pi] \to \mathbb{C}$

$t \mapsto \varepsilon e^{it}$

$$= \underbrace{\int_{c_R} \frac{e^{ibz}}{z} dz}_{= 0} + \int_0^{\pi} \underbrace{\frac{e^{ib\varepsilon e^{it}}}{\cancel{\varepsilon e^{it}}}}_{\xrightarrow{\varepsilon \to 0} 1} i \cancel{\varepsilon e^{it}} \, dt = \underline{\underline{\pi i}}$$

Falls $b < 0$ muss in <u>unteren</u> Halbebene geschlossen werden:

$J(b) =$  $+$  $= -\pi i$

d.h. $$\boxed{J(b) \equiv \int_{-\infty}^{\infty} \frac{e^{ibu}}{u}\, du = \pi i \operatorname{sgn} b}$$