Information Theory & Statistical Physics

Lecture: Prof. Dr. Johannes Berg
Exercises: Stephan Kleinbölting **Sheet 4**

Due date: 23.06.17 12:00 *To be discussed on: 28.06.17*

Website: http://www.thp.uni-koeln.de/~skleinbo/teaching/Information2017/

13 Dissipating logic 25 pts.

Landauer's principle states that any irreversible logical computation must be implemented by an irreversible physical process. For otherwise one might reverse the physical process and obtain the inputs from the outputs. This is by definition impossible for an irreversible computation.

(a) *15pt -* Consider a logic gate that maps three input bits P, Q, R to

$$(P, Q, R) \mapsto (\overline{P \wedge Q},\ R,\ P \vee Q)$$

and assume each input state appears with equal probability.[1]

Write down the truth-table of the gate! How much heat is dissipated on average?

Hint: Think about how heat is related to the entropies of in- and output states.

(b) *10pt -* On the other hand Landauer's principle implies that a reversible logic gate may operate in principle without heat dissipation (at least from information erasure). The *Toffoli gate* is an example of a reversible gate. In particular it is also universal, i.e. any logic operation can be build using only the Toffoli gate. It's described by the mapping

$$(P, Q, R) \mapsto (P,\ Q,\ R \veebar (P \wedge Q))$$

Repeat your analysis from part (a) and show that no heat is dissipated on average. Again, the inputs are equiprobable.

14 Kraft-McMillan II 25 pts.

You have seen in the lecture as well as in problem no. 10 that any finite D-ary *prefix* code necessarily fulfills the Kraft inequality

$$\sum_x D^{-l(x)} \leq 1.$$

Surprisingly it turns out that not only prefix codes fulfill this inequality, but actually all uniquely decodable codes do! Let us proof this more general statement[2].

(a) *20pt -* Consider a uniquely decodable code (UDC) C over a finite alphabet $\mathcal{X}$. Recall from problem 10 on sheet 2 the definition of the extended code C^+ as the code that maps strings of source symbols onto strings of code symbols. By definition of unique decodability no two strings have the same encoding. Note that the extended code has lengths

$$l(x_1 x_2 \cdots x_k) = \sum_{j=1}^{k} l(x_j).$$

[1] $\wedge$ is the logical AND; $\vee$ means OR; an over-bar negation; $\veebar$ is XOR

[2] The extension to uniquely decodable codes was proven not by Kraft but by McMillan(1956). It is therefor more appropriate to speak of the Kraft-McMillan-inequality.

Prove the Kraft-McMillan-inequality for finite uniquely decodable codes!

Hint: Start by considering

$$\left(\sum_x D^{-l(x)}\right)^k$$

for some positive integer k and bound this expression from above. It might be a good idea to think about how many code strings of length k there maximally are.

(b) *5pt* - Argue that uniquely decodable codes do not improve on prefix codes when it comes to average length.

This is somewhat surprising since the set of prefix codes is strictly smaller than that of uniquely decodable ones. It also implies that for any UDC one can find a prefix code that performs just as well.

15 Huffman Codes 10 pts.

A binary Huffman code is generated by successively coalescing the symbols with the smallest probabilities.

It was claimed in the lecture that (binary) Huffman codes are optimal. We would like to proof this.

The key step in the algorithm is assigning codewords of equal length to the two least probable symbols. Let us show by contradiction that this strategy is indeed optimal.

Let a, b denote the two least probable symbols. Assume we had an optimal and uniquely decodable code C such that $l(a) < l(b)$.

In light of the previous problem, argue that this code cannot be optimal!