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1 Residues (4×2=8 P)

Determine the residues at the poles of the following functions

a.)

$$f_1(z) = \frac{1}{z(z-a)^3}$$

b.)

$$f_2(z) = \frac{\sin z}{z}$$

c.)

$$f_3(z) = \frac{e^z}{\sin \mathrm{i}z}$$

d.)

$$f_4(z) = z^3 \exp\left(\frac{1}{z^2}\right)$$

2 Roots and pole-counting integral (6 P)

Set

$$\operatorname{ord}(f, z_0) := \begin{cases} k & f \text{ has a root of order } k \text{ at } z_0 \\ -k & f \text{ has a pole of order } k \text{ at } z_0 \\ 0 & \text{else} \end{cases}$$

Let $f : U \to \mathbb{C}$ be holomorphic with a root of order k at $z_0 \in U$.
Show that

$$\operatorname{Res}\left[\frac{f'}{f}, z_0\right] = k$$

Hint: Taylor-expand around the root.
Now consider f having a pole of order k at z_0. Show that

$$\operatorname{Res}\left[\frac{f'}{f}, z_0\right] = -k$$

Argue that if f has roots N and poles P in U, and Γ is a path in U that encircles all poles and roots, then

$$\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{w \in N \cup P} \operatorname{ind}_\Gamma(w) \operatorname{ord}(f, w)$$

where $\operatorname{ind}_\Gamma(w)$ is sense of circulation ("*Index*") of the path around a point w.
That mean, if Γ is a path running around each pole and root exactly once and in counter-clockwise direction, the integral counts the difference between roots and poles weighted by their respective order.

3 Green Function II (10 P) In problem 2 of sheet 7 we developed the Green function of the harmonic oscillator up to evaluation of the integral

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}\omega(t-t')}}{-\omega^2 - 2\mathrm{i}\gamma\omega + \omega_0^2}\,\mathrm{d}\omega$$

Using the calculus of residues, we are now in a position to calculate this integral. Assume for now that $\omega_0 \neq \gamma$, but do not restrict yourselves to $\omega_0 > \gamma$. You may need to distinguish cases.

a) (2 P) Convert the integral into a complex line integral along the real axis.

b) (1 P) Make a drawing of the complex plane and indicate the poles and integration path.

c) (2 P) We need a closed contour in order to apply the residue theorem.

Close the contour with an additional path of your choice that does not contribute to the integral. Explain your choice.

d) (3 P) Evaluate the integral along the now closed loop.

Compare your result to the expression in problem 2c) on sheet 7. Does yours contain the theta-function too? If not, go back to part b).

e) (2 P) Consider now the case of critical damping $\omega_0 = \gamma$ and evaluate the integral anew for this case.

4 Infinite series and residues (8+3 P) We'd like to determine the value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

a) (2 P) Determine the residues of $f(z) = \cot \pi z \equiv \dfrac{\cos \pi z}{\sin \pi z}$ at its poles.

b) (1 P) Let U be a simply connected neighbourhood of z_0. Let $f : U \setminus \{z_0\} \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be holomorphic. Let f have a first-order pole at z_0. Show that

$$\operatorname{Res}[gf, z_0] = g(z_0)\operatorname{Res}[f, z_0]$$

c) (2 P) Determine the residues of $\dfrac{\cot \pi z}{z^2}$.

d) (3 P) Let $M \in \mathbb{N}$. Consider the square path Γ_M, fixed by the sequence of its corner points

$$(M+\frac{1}{2})(1+\mathrm{i}) \longrightarrow -(M+\frac{1}{2})(1-\mathrm{i}) \longrightarrow -(M+\frac{1}{2})(1+\mathrm{i}) \longrightarrow (M+\frac{1}{2})(1-\mathrm{i})$$

Draw the path in the complex plane and mark the poles of $\dfrac{\cot \pi z}{z^2}$.

Show that

$$\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_M} \frac{\cot \pi z}{z^2}\,\mathrm{d}z = -\frac{\pi}{3} + \frac{2}{\pi}\sum_{n=1}^{M}\frac{1}{n^2}$$

We are after the limit $M \to \infty$. Give an upper bound of the integral on the left-hand-side for large M and show that it vanishes in the limit $M \to \infty$ (cf. Sheet 8, problem 3d).

What is

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

then?

e) (+3 P) What is the value of $\sum_{n=1}^{\infty} n^{-4}$?