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1 Linear maps and their matrix representations (10 P) Determine whether the following maps are linear. If they are, obtain the matrix wrt. the indicated bases.

$\mathbb{R}^n$ is equipped with component-wise addition and multiplication by a real scalar.

Sets of functions often admit a vector space structure. A basis of such a function space is then a set of functions from which one may linearly combine any other function. In the following we regard the particularly accessible space of polynomials.

Addition and scalar multiplication are declared pointwise:

$$+ : (p, q) \mapsto (x \mapsto p(x) + q(x))$$

$$\cdot : (\lambda, p) \mapsto (x \mapsto \lambda p(x))$$

gegeben.

We define bases $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, $\mathcal{B}' = \{\vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2\}$, $\mathcal{P}_n = \{x \mapsto 1, x \mapsto x, x \mapsto x^2, \ldots, x \mapsto x^n\}$. (For short $\mathcal{P}_n = \{1, x, x^2, \ldots, x^n\}$).

a) (3 P) $F : \mathbb{R}^3 \to \mathbb{R}^2 : \; (x_1, x_2, x_3) \mapsto (3x_1 - x_3, x_1 + x_2 + x_3)$. Give the representation wrt. the bases $\mathcal{B}, \mathcal{B}'$.

b) (4 P) Let P_n the space of real polynomials of degree less or equal n and consider the map $G : P_n \to P_n : \; (x \mapsto p(x)) \mapsto (x \mapsto x^m p'(x))$. Let $m \in \mathbb{N}_0$. Give the representation wrt. the basis $\mathcal{P}$.

c) (3 P) $T_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n : \; \vec{x} \mapsto \vec{x} + \vec{a}$ with $\vec{a} \in \mathbb{R}^n$. Give the representation wrt. the canonical basis.

2 Kernel and image (10 P) The notions of *kernel* and *image* of a linear map $L : V \to W$ between vector spaces were introduced in the lecture. It was demonstrated that the kernel is a subspace of V.

a) (3 P) Proof that $\operatorname{im} L \subset W$ is a subspace of W.

b) (3 P) Determine the kernel of F from problem 1a. What is the dimension of the image of F? What is the dimension of kernel and image of the map G from 1b) for $m \in \{0, 1\}$?

c) (4 P) The kernel plays an important role in determining preimages of linear maps, and hence for the solutions of inhomogenous, linear (differential) equations.

Reminder: The set $L^{-1}(w) = \{v \in V \big| Lv = w\}$ is called *preimage of* w.

Proof the following statement: If $L : V \to W$ is linear and $w \in W$, then the preimage of w is given by

$$L^{-1}(w) = u + \ker L := \{u + v \big| v \in \ker L\}$$

where $u \in V$ is *any* vector such that $L(u) = w$.

3 Matrix product (10 P) Define

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 7 & -2 \\ -3 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

a) (4 P) Calculate

$$\mathbf{A} \cdot \mathbf{B}, \mathbf{B} \cdot \mathbf{A}, \mathbf{B} \cdot \mathbf{C}, \mathbf{C} \cdot \mathbf{B}, \mathbf{C} \cdot \mathbf{C}$$

if possible. Between spaces of which dimensions do they map?

b) (3 P) Let

$$T : \mathbb{R}^2 \to \mathbb{R}^3, (x_1, x_2) \mapsto (x_1 - x_2, x_1 + x_2, 0)$$

and

$$S : \mathbb{R}^3 \to \mathbb{R}^2 : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3, -(x_1 + x_2 + x_3))$$

Calculate the matrix representations of T, S and $S \circ T$ wrt. the canonical bases and verify that

$$\mathbf{M}(S)\mathbf{M}(T) = \mathbf{M}(S \circ T)$$

c) (3 P) Consider P_4 from 1b). Determine the matrix of the second derivative $\mathbf{M}(\frac{\mathrm{d}^2}{\mathrm{d}x^2})$.