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1 Rotations in the plane (10 P) Consider the plane $\mathbb{R}^2$. Rotations around the origin about an angle φ are generated by a map

$$R(\varphi) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \varphi \in [0, 2\pi)$$

a) (2 P) Argue that $R(\varphi)$ is linear.

b) (3 P) Make a drawing and find the matrix representation $\mathbf{R}(\varphi)$ wrt. the canonical basis $\{\vec{e}_1, \vec{e}_2\}$.

c) (3 P) Proof that the set of rotation matrices $\mathbf{R}(\varphi)$ is an abelian group under matrix multiplication.

Note: The group of 2x2 rotation matrices is called the *special orthogonal group*, short SO_2. We are going to investigate later what that designation means.

d) (2 P) Consider now $\mathbb{R}^3$. Apparently there are now many more possible rotation axes. Using the result from b), find the matrices $\mathbf{R}_x(\varphi), \mathbf{R}_y(\varphi), \mathbf{R}_z(\varphi)$ that represent rotations around the x, y, z-axes.

2 The Gauß-Jordan-algorithm (20 P) In this problem we are going to develop an algorithm that allows us to solve arbitrary systems of linear equations as well as find inverses of invertible matrices.

Consider linear systems first. Let $\mathbf{A} \in \text{Mat}(m \times n)$ be a $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Consider the system

$$\mathbf{A}\mathbf{x} = \mathbf{b} \text{ equivalent to } \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\text{or in components} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Introduce

$$(\mathbf{A}, \mathbf{b}) := \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

as the *extended coefficient matrix.* It contains all information about the system of equations.

The first (and most important step) of the algorithm consists in transforming the extended coefficient matrix into *row-echelon-form* (REF). A matrix is in REF, if the number of leading zeroes in each row is strictly increasing from top to bottom (cf. fig. 1).

This can be achieved by successive application of the following elementary row reductions

i) Exchange of two rows.

ii) Multiplying a row by a non-zero scalar.

iii) Adding a multiple of the j-th row to the i-th row.

Let $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}})$ be a coefficient matrix that was obtained by a reduction operation from $(\mathbf{A}, \mathbf{b})$.

a) (3 P) Show that the space of solutions is invariant under row reductions. I.e. if $\mathbf{x} \in \mathbb{R}^n$ fulfills $\mathbf{A}\mathbf{x} = \mathbf{b}$, then $\tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}$ is fulfilled too. Argue then that arbitrary concationations of row reductions are permitted without altering the solution space.

b) (3 P) The operations i)–iii) are generated by left multiplication with so called *elementary matrices*

$$\tilde{\mathbf{A}} = \mathbf{E}\mathbf{A}$$

Find those matrices and demonstrate that they are isomorphisms (don't change the rank of $\mathbf{A}$).

Row-echelon-form is achieved through the following steps:

G1) Sort the rows by their number of leading zeros. One obtains a matrix of the form

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & & & & b_1 \\ 0 & a_{22} & \cdots & & & \vdots \\ 0 & a_{32} & \cdots & & & \\ 0 & 0 & 0 & a_{44} & \cdots & \\ 0 & 0 & 0 & a_{54} & \cdots & \\ 0 & 0 & 0 & 0 & a_{66} & \vdots \\ 0 & & \cdots & & 0 & b_7 \end{array}\right)$$

G2) Starting at the bottom row, try to eliminate the leading coefficient of any row a_i. If the row above a_{i-1} has its leading non-zero coefficient at the same position, subtract (operation iii.) a suitable multiple from a_i.

$$a_i \mapsto a_i - \lambda a_{i-1}$$

This step is iterated from bottom to top until there are no more coefficients to eliminate. The resulting matrix looks like (fig. 1)

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & & & & b_1 \\ 0 & a_{22} & \cdots & & & \vdots \\ 0 & 0 & 0 & a_{34} & \cdots & \\ 0 & 0 & 0 & 0 & a_{45} & \\ 0 & & \cdots & & 0 & \vdots \\ 0 & & \cdots & & 0 & b_7 \end{array}\right)$$

Figure 1: Matrix in row-echelo-form. The leading coefficients are non-zero. The number of leading zeros is strictly increasing from top to bottom.

c) (4 P) Can you tell from the REF if a solution exists? Can you tell wether the solution is unique and if not, what the dimension of the solution space is going to be?

One may read of the solution from the REF.

We are going to extend the algorithm a little. Most important application of this so called Gauß-Jordan-reduction is in determining inverse matrices and finding determinants (later).

GJ3) Divide each row by its leading coeff and set it to 1.

GJ4) Apply row reductions such that every *column* that contains a leading coefficient has zeros everywhere else. (Convince yourself that this is possible and how the resulting matrix looks like!)

Following these steps leads to the *reduced REF* (rREF).

d) (4 P) Determine the solutions of the following systems by bringing the coefficient matrices into reduced REF.

(i)

$$\left(\begin{array}{cccc|c} 3 & 1 & 21 & 13 & 15 \\ 0 & 1 & 0 & 0 & -4 \\ 6 & 2 & 43 & 26 & 30 \\ 3 & 1 & 18 & 12 & 14 \end{array}\right)$$

(ii)

$$\left(\begin{array}{cccc|c} 0 & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 \end{array}\right)$$

e) (3 P) Show that: A $n \times n$ matrix is invertible iff[1] its rREF is the unit matrix.

Demonstrate further that the inverse may be calculated by writing

$$\left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & & \\ \vdots & & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} & & & 1 \end{array}\right)$$

and applying the same operations left and right, such that the left hand side is transformed into the unit matrix. The right hand side is then the inverse.

f) (3 P) Find the inverse of

$$\begin{pmatrix} \frac{3}{2} & 13 & 3 & 0 & 3 \\ 0 & 0 & \frac{9}{2} & 1 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & 0 & 0 \\ \frac{1}{2} & 5 & 0 & 0 & 1 \end{pmatrix}$$

[1] iff: if and only if