Lineare Algebra und Vektoranalysis
Sommersemester 2019
Blatt 3, Abgabe 25.04.2019 bis 10:00

Institut für Biologische Physik
J. Berg
U. Michel, S. Kleinbölting

1 Determinants (22 P)

You have learned about two explicit techniques for calculating determinants: Leibniz' rule and Laplace's rules.

The Leibniz method isn't particularly effecient since the number of terms grows as $n!$. It is better suited for proofs and is often used as a definition of the determinant.

Expanding the determinant along a row or column has basically the same problem. In the first step one needs to calculate n sub-determinants For each of those then $n-1$ and so on. This is managable for matrices of size 3,4 or 5, particularly if they contain rows or columns with many zeros.

a) (6 P) The Gauß-(Jordan)-algorithm implies a much more efficient method for calculating determinants. It uses the fact that the determinant of a triangular matrix is given by the product of its diagonal.

Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}$$

be an upper triagonal matrix, i.e. all entries below the diagonal are zero. Proof that

$$\det \mathbf{A} = \prod_{i=1}^{n} a_{ii}$$

Is this valid for lower triagonal matrices as well?

Describe a method for calculating the determinant by bringing the matrix into row-echelon-form. Remember the properties of determinants under row reductions.

Reminder: $\det \mathbf{A} = \det \mathbf{A}^t$.

b) (4 P) Calculate the determinants of

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -3 & 1 \\ 2 & -1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a & a & a & 0 \\ a & 0 & 0 & b \\ 0 & 0 & b & b \\ a & b & b & 0 \end{pmatrix}$$

c) (3 P) *Sarrus' rule* is a simple scheme for calculating determinants of 3x3 matrices. One begins by repeating the matrix right next to the original. One then forms products along the diagonals (each diagonal is a product of three entries).

The diagonal products are added. The ones in $\searrow$-direction come with a "+" sign, while the ones in $\swarrow$-direction get a "-" (cf. figure).

Show that Sarrus's rule is equivalent to Laplace's rule.

[1]Bildnachweis: Eisenbahn%s [CC BY-SA 4.0 (`https://creativecommons.org/licenses/by-sa/4.0`)]

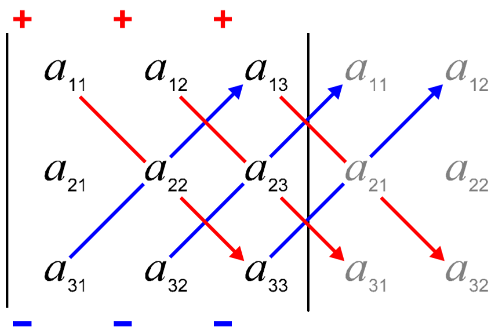

Figure 1: Regel von Sarrus[1]

d) (2 P) The following properties hold in general for $n \times n$-matrices $\mathbf{X}, \mathbf{Y}$:

$$\text{(i)} \quad \det(\mathbf{XY}) = \det(\mathbf{X})\det(\mathbf{Y}) \qquad \text{(ii)} \quad \det(\mathbf{X}^{-1}) = \frac{1}{\det(\mathbf{X})}$$

Motivate those properties from geometric considerations about X, Y, and X^{-1} respectively.

e) (2 P) Let $\mathbf{A} \in \mathrm{Mat}(n \times k), \mathbf{B} \in \mathrm{Mat}(k \times n)$ and $k < n$. What can you deduce about the determinant $\det \mathbf{AB}$ and why?

f) (5 P) In the lecture it was shown that the eigenvalues of a matrix are the roots of the characteristic polynomial

$$\chi_{\mathbf{A}}(\lambda) := \det(\mathbf{A} - \lambda \mathbf{I}_n)$$

Write a short(!) essay that recapitulates the proof.

Keywords: Kernel, invertible, eigenvalue, eigenvector, eigenspace.

2 Equivalence of row and column rank (8 P) In the lecture the *rank* of a map/matrix was defined as the dimension of the image or the number of linearily independent columns equivalently.

How about the dimension of the row space? It's not a priori clear that those numbers are the same. We will prove this now.

The *row- / column-rank* of a matrix is the number of linearily independent rows/columns.

a) (2 P) Let $\mathbf{A} \in \mathrm{Mat}(m \times n)$ be an arbitrary matrix in (reduced) row-echelon-form. Demonstrate that row- and column rank are equal.

b) (6 P) We proved in 2.2 that row reductions leave the column-rank invariant.

We only need to show that the row-rank is invariant as well. Let $(a_1, \ldots, a_m)$ be the rows of the matrix $\mathbf{A}$ and $(\tilde{a}_1, \ldots, \tilde{a}_m)$ the rows of the row-reduced matrix $\tilde{\mathbf{A}}$.

We can even show the stronger statement: The row space is invariant under row reductions.

Show: $x \in \mathbb{R}^m$ is in the space spanned by $(a_1, \ldots, a_m)$ iff it's in the span of $(\tilde{a}_1, \ldots, \tilde{a}_m)$.

c) (1 P) Finally argue that row-rank=column-rank holds for any matrix. We call it simply *rank* from now on.

3 Matrix-Normalform (Bonus +5 P)

By applying row-reductions we may transform any matrix into a convenient form. On the other hand we might as well perform column reductions. The reduced REF may be simplified even further by eliminating "extraneous" columns.

In 2.2 we introduced the elementary matrices. Row reductions are generated by multiplication from the left with these matrices.

For the sake of completeness we list them here (2).

$$\mathbf{S}_i(\lambda) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \mathbf{Q}_i^j(\lambda) = \begin{pmatrix} 1 & & & & \\ & 1 & & \lambda & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \quad \mathbf{P}_i^j = \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix}$$

(a) Multiplication of the i-th row with λ

(b) Adding the λ-fold of the j-th to the i-th row.

(c) Exchanging i-th and j-th row.

Figure 2: Elementare Zeilenumformungen

a) (2 P) Show that elementary column reductions are generated by multiplication *from the right.* This also leaves the rank invariant.

b) (3 P) Argue that one can transform any matrix $\mathbf{A}$ through a series of row- and column-reductions

$$\mathbf{A} \mapsto \mathbf{R}_j \cdots \mathbf{R}_1 \mathbf{A} \mathbf{C}_1 \cdots \mathbf{C}_k$$

into the normal-form

$$\left(\begin{array}{c|c} \mathbf{I}_r & \mathbf{0}_{r,n-r} \\ \hline \mathbf{0}_{m-r,r} & \mathbf{0}_{m-r,r} \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 1 & & & 0 \\ \hline 0 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 0 \end{array}\right)$$

$\mathbf{I}_r$ is the $r \times r$ unit-matrix. What is r?

Wir wünschen schöne Ostertage!