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1 Eigenvalues/-vectors and diagonalization (12 P) Determine the eigenvalues and all corresponding eigenvectors of the following matrices.

a)
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ -1 & 2 & 1 \end{pmatrix},$$

b)
$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

c)
$$\mathbf{C} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

d)
$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & b^* & 0 & 0 \\ a^* & 0 & 0 & 0 \end{pmatrix}$$

Interpret your results for $\mathbf{B}, \mathbf{C}$ geometrically.

Argue whether each matrix is diagonalizable. If yes, write down the corresponding change of basis matrix and verify by explicit multiplication that each matrix is appropriately diagonalized.

2 Matrix exponential (18 P) Let $\mathrm{Mat}(n \times n, \mathbb{K})$, mit $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be the set of all real- or complex-valued $n \times n$-matrices and $\mathbf{I}_n \in \mathrm{Mat}(n \times n, \mathbb{K})$ the $n \times n$-identity-matrix.

Since we know how to form polynomials of matrices, one could try to extend real or complex power serieses to matrices and therefore make a lot of useful functions accessible to matrices.

As motivation and because it is important in practice we consider the exponential function. Analagously to the complex exponential we define for $\mathbf{A} \in \mathrm{Mat}(n \times n, \mathbb{K})$:

$$\exp(\mathbf{A}) = \mathbf{I}_n + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{A}^k$$

a) (2 P) As was demonstrated in the lecture, a system of linear differential equations can be cast into matrix form.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{1}$$

Assume $\mathbf{A}$ to be constant (i.e. its entries do not depend on t). Demonstrate that

$$\mathbf{x}(t) = \exp(t\mathbf{A})\mathbf{x_0}$$

is a solution of (1) with initial condition $\mathbf{x}(0) = \mathbf{x_0}$.

Hint: One can differentiate the exponential term by term.

Solving a linear ODE (with constant coefficients) hence reduces to finding a matrix exponential. Reason enough to study them!

b) (4 P) Calculate the exponentials of the following matrices:

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_4 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

In these four examples, the exponential could be calculated relatively easily by brute force. This is not always possible. If a matrix is diagonalizable, there is a more systematic approach.

c) (1 P) Let $\mathbf{D} = \operatorname{diag}(d_1, \ldots, d_n) \in \operatorname{Mat}(n \times n, \mathbb{K})$ be an arbitrary $n \times n$ diagonal matrix. Show that

$$\exp(\mathbf{D}) = \operatorname{diag}\left(\mathrm{e}^{d_1}, \ldots, \mathrm{e}^{d_n}\right) \tag{2}$$

d) (1 P) Let $\mathbf{X} \in \operatorname{Mat}(n \times n, \mathbb{K})$ be diagonalizable, i.e. there are matrices $\mathbf{P}, \mathbf{D}$, such that $\mathbf{X} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal. Deduce that

$$\exp(\mathbf{X}) = \mathbf{P}\exp(\mathbf{D})\mathbf{P}^{-1},$$

gilt, where $\exp(\mathbf{D})$ is given by (2).

e) (4 P) Use the formula from d) to determine the exponential of $\mathbf{A}$ und $\mathbf{D}$ from problem 1.

f) (4 P) Expression (1) is valid only for first-order ODEs. This doesn't pose a substential obstacle though, because any equation of higher order can be written as a system of first-order equations.

Consider again the damped harmonic oscillator $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$. Introducing an auxiliary variable $v \equiv \dot{x}$, allows us to cast the ODE in the form

$$\begin{pmatrix} \dot{v} \\ \dot{x} \end{pmatrix} = \overbrace{\begin{pmatrix} -\gamma & -\omega_0^2 \\ 1 & 0 \end{pmatrix}}^{\mathbf{A}} \begin{pmatrix} v \\ x \end{pmatrix}$$

Verify this! Find the solution to initial conditions (v_0, x_0) by using results from a) and d).

g) (2 P) Matrix exponentials do not behave like complex exponentials and one needs to exercise caution. For example the regular exponential function satisfies

$$\mathrm{e}^a\mathrm{e}^b = \mathrm{e}^{a+b} = \mathrm{e}^b\mathrm{e}^a.$$

Show that the matrix exponential does *not* fulfill this identity in general. Can you guess why that might be?

Hint: Find a simple counterexample.