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1 Transformation behaviour under a change of basis (20 P) The object we deal with in linear algebra, i.e. vectors, linear maps, are all declared independetly of a choice of basis. This needs to be so, since a choice of basis is completely arbitrary. Thus the component representations of these objects (column vectors, matrices,...) are also arbitrary. In practice one often tries to find a basis in which the components take on a particularly simple form (e.g. a diagonal matrix).

In order to be able to describe the same object in different bases, one must know how their components change under a change of basis. All transformation properties are direct consequences of this invariance. The vector or the linear map themselves do not depend on the choice of basis.

Let V and W be vectorspaces over $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{E} = \{e_i\}, \mathcal{E}' = \{e'_i\} \subset V$ and $\mathcal{F} = \{f_i\}, \mathcal{F}' = \{f'_i\} \subset W$ respectively arbitrary bases of these spaces. In particular there exist coefficients S_{ij} and T_{ij} such that

$$e_i = \sum e'_j S_{ji}$$
$$f_i = \sum f'_j T_{ji}$$

Let further $L : V \to W$ be a linear map.

a) (1 P) Deduce from the invariance of an arbitrary vector $v \in V$, that its components transform according to

$$\mathbf{v}' = \mathbf{S}\mathbf{v}$$

under a change of basis.

b) (1 P) Deduce further from the invariance of $w = Lv$, that by changing bases in V as well as W, the matrix representation of L transforms according to

$$\mathbf{L}' = \mathbf{T}\mathbf{L}\mathbf{S}^{-1}.$$

We are now going to investigate an object that transforms neither like a vector nor a linear map. The lecture introduced the notion of *scalar (or inner) product.* We restrict ourselves to the case of a real, euclidean inner product.

As you are aware, an inner product is a *bi*linear map $V \times V \to \mathbb{R}$. It also exists independently from a choice a basis.

We may still calculate the inner product from the components of the involved vectors after choosing a (not necessarily orthonormal) basis by writing

$$\langle u, v \rangle = \left\langle \sum_i u_i e_i, \sum_j v_j e_j \right\rangle = \sum_{i,j} u_i v_j \langle e_i, e_j \rangle \equiv \sum_{i,j} u_i v_j g_{ij} \tag{1}$$

The last equality defines the quantity $g_{ij} = \langle e_i, e_j \rangle$ which is called the *metric tensor.* It contains in a given basis all information about an inner product – just like matrices define a linear map completely.

c) (2 P) Convince yourself that (1) can be written in the form

$$\langle u, v\rangle = \mathbf{u^t g v} \tag{2}$$

, where $\mathbf{u^t} = (u_1, \ldots, u_n)$ is a row vector and $(\mathbf{g})_{ij} = g_{ij}$ the matrix of components of the metric tensor.

d) (3 P) Proof that with respect to an orthonormal basis

$$\langle u, v\rangle = \mathbf{u^t I v} = \mathbf{u} \cdot \mathbf{v} = \sum u_i v_i \tag{3}$$

$\mathbf{I}$ is the identity matrix. Hence any inner product is the standard dot product in an orthonormal basis.

What happens under a change of basis?

Show that (3) is in general *not* invariant under a change of basis (e.g. by scaling the basis). Which property does a change of basis matrix need to have, in order to leave $\mathbf{u} \cdot \mathbf{v}$ invariant?

To define an invariant object it is therefore not sufficient to only regard $\mathbf{u} \cdot \mathbf{v}$. One needs to transform the metric tensor too. Again, the transformation behaviour follows from demanding invariance of the inner product under change of basis.

e) (5 P) Deduce from the invariance of $\langle u, v\rangle$ under basis change that the metric tensor transforms as

$$\mathbf{g}' = (\mathbf{S^{-1}})^{\mathbf{t}} \mathbf{g} \mathbf{S^{-1}}$$

if $\mathbf{u}' = \mathbf{S u}$. Show first that transposition and inversion commute, i.e. that

$$(\mathbf{S^{-1}})^{\mathbf{t}} = (\mathbf{S^t})^{-\mathbf{1}}$$

[*Hint:* Show and use $(\mathbf{AB})^{\mathbf{t}} = \mathbf{B^t A^t}$ first].

f) (5 P) Proof further that the defining properties of an inner product imply the following properties of $\mathbf{g}$

(i.) $\mathbf{g}$ is symmetric.

(ii.) $\mathbf{g}$ is invertible.

(iii.) $\mathbf{g}$ is diagonalizable and its eigenvalues are strictly positive

and that vice versa any symmetric matrix with strictly positive eigenvalues defines an inner product according to (2)

g) (3 P) Let $\mathbf{g} = \begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}$ wrt. the canonical basis of $\mathbb{R}^2$. For which $a \in \mathbb{R}$ does this define a valid inner product?

Find the basis that is orthonormal wrt. to the inner product given by $\mathbf{g}$. Diagonalize $\mathbf{g}$ first and then think about how you can achive normalization of the basis vectors.

2 Orthogonal Maps (10 P) In the lecture you learned about *orthogonal maps.* A linear map of an euclidean vector space $L : V \to V$ on itself[1] is called *orthogonal* if

$$\forall x, y \in V : \langle Lx, Ly \rangle = \langle x, y \rangle$$

holds.

Call $O(V)$ the set of all orthogonal maps on V.

We are going to discuss some important properties of these maps.

a) (3 P) Proof that L is invertible [2] and that L^{-1} is orthogonal as well.

Further show that $O(V)$ is a group under concatination.

What is there to say about matrix representations of orthogonal maps?

b) (2 P) Show that a matrix $\mathbf{L}$ of $L \in O(V)$ wrt. an orthonormal basis fulfills

$$\mathbf{L}^{-1} = \mathbf{L}^t$$

We call a matrix *orthogonal* if it has this property.

c) (3 P) Orthogonal matrices may be defined by many equivalent properties.

Show that the following are equivalent

(i) $\mathbf{L}^{-1} = \mathbf{L}^t$.

(ii) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \mathbf{Lx} \cdot \mathbf{Ly} = \mathbf{x} \cdot \mathbf{y}$.

(iii) The columns $(\mathbf{l}_1, \ldots \mathbf{l}_n)$ of $\mathbf{L}$ are orthonormal: $\mathbf{l}_i \cdot \mathbf{l}_j = \delta_{ij}$.

d) (2 P) Proof that $\det(L) = \pm 1$. Which kind of transformations do the different signs represent? Consider the special case of $\mathbb{R}^3$ for that.

[1] called an *endomorphism*

[2] an invertible linear map is called an *isomorphism*