The *real Fourier series* of a $2T$-periodic function $f : \mathbb{R} \to \mathbb{R}$ ($f(x+2T) = f(x)\ \forall x \in \mathbb{R}$) is defined as:

$$\begin{aligned} f(x) =& \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x)\right) \quad \text{with} \qquad (1)\\ \omega =& \frac{\pi}{T}, \quad \frac{a_0}{2} = \frac{1}{2T}\int_0^{2T} f(x)\mathrm{d}x,\\ a_n =& \frac{1}{T}\int_0^{2T} f(x)\cos(n\omega x)\mathrm{d}x, \quad b_n = \frac{1}{T}\int_0^{2T} f(x)\sin(n\omega x)\mathrm{d}x. \end{aligned}$$

1 Properties of the Fourier coefficients (14 P)

a) (2 P) Show that if $f(x)$ has period $2T$, then for any $c \in \mathbb{R}$

$$\int_0^{2T} f(x)\mathrm{d}x = \int_c^{c+2T} f(x)\mathrm{d}x$$

Argue that one is free to integrate over any interval of length T to calculate the Fourier coefficients a_n, b_n.

Hint: Partition the integration domain and use the periodicity of the integrand.

This often allows to simplify calculations by chosing the integration domain cleverly.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $2T$-periodic function. A function is called *even*, if

$$f(-x) = f(x) \quad \forall x \in \mathbb{R},$$

and *odd*, if

$$f(-x) = -f(x) \quad \forall x \in \mathbb{R}.$$

b) (2 P) Proof that $f(x) = \cos(\alpha x)\ \forall \alpha \in \mathbb{R}$ is even.
Hint: Euler's formula.

c) (2 P) Proof that $g(x) = \sin(\beta x)\ \forall \beta \in \mathbb{R}$ is odd.

d) (4 P) Show that for any odd function $f_u : \mathbb{R} \to \mathbb{R}$, and even function $f_g : \mathbb{R} \to \mathbb{R}$ respectively:

$$\int_{-a}^{a} f_u(x)\mathrm{d}x = 0 \quad \forall a \geq 0 \quad \text{and} \quad \int_{-b}^{b} f_g(x)\mathrm{d}x = 2\int_0^b f_g(x)\mathrm{d}x \quad \forall b \geq 0.$$

e) (4 P) Proof the following statements

a) The real Fourier series (1) of an odd, $2T$-periodic function f_u has $a_0 = 0$ and $a_n = 0\ \forall n \in \mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

b) The real Fourier series (1) of an even, $2T$-periodic function f_g has $b_n = 0$ $\forall n \in \mathbb{N}_+$.

2 Fourier serieses of periodic functions (16 P)

a) (2 P) Draw the *saw-tooth function* defined as $f(x) = -2x$ on the interval $]-2, 2]$ and periodically continued.

b) (10 P) Determine the real Fourier series of said saw-tooth.

Hint: You may plot your result on `www.wolframalpha.com`. For example: the command

`plot 4/Pi*(sum 1/(2*n+1)*sin((2*n+1) x) from n=0 to 9),`

draw the Fourier series of the rectangular function from the lecture up to order 9.

c) (4 P) Determine the Fourier series of $f(x) = \sin^2(6x)$ for half-period $T = \pi$.

Hint: You do not need to calculate any integrals.

3 Complex vs. real Fourier series (10 P) Analogously to the real Fourier series (1) one defines the *complex Fourier series* of real-valued, $2T$-periodic functions $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\omega x} \quad \text{with } \omega = \frac{\pi}{T} \text{ und } c_n = \frac{1}{2T} \int_0^{2T} f(x)\mathrm{e}^{-in\omega x}\mathrm{d}x \ \forall n \in \mathbb{Z}$$

a) (7 P) Proof the following connection between real and complex Fourier coefficients

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n + ib_n) \ \forall n < 0, \quad c_n = \frac{1}{2}(a_n - ib_n) \ \forall n > 0.$$

b) (3 P) Calculate the complex coefficients for the saw-tooth function from problem 2.