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From now on we are going to use the notation

$$\langle f, g \rangle \equiv \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x$$

for arbitrary functions f, g as long as the integral exists.

1 The delta-"function" (7 P) The lecture introduced the delta-function as the limit of a sequence of Gaussians with ever decreasing width. Its sieve-property

$$\langle \delta, f \rangle = \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x = f(0)$$

was demonstrated.
Let $f : \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function that vanishes at infinity.

a) (1+1+2+1 P) Show the following properties of the delta-function through appropriate substitutions and integrations by part.

i.) We write $\delta_a(x) \equiv \delta(x - a)$. The delta-function has the translation property

$$\langle \delta_a, f \rangle = f(a)$$

ii.) It behaves under scaling as

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

iii.) Let g be a differentiable function with isolated, simple roots $\{x_i\}$. Then

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

Use this to determine ($x_0 \neq 0$)

$$\int f(x)\delta(x^2 - x_0^2)\mathrm{d}x$$

Hint: Expand f, g in the vicinity of any root of g to first order.

iv.) Derivatives of the delta-function ($f^{(n)} \equiv \frac{\mathrm{d}^n f}{\mathrm{d}^n}$)

$$\left\langle \delta^{(n)}, f \right\rangle = (-1)^n f^{(n)}(0)$$

b) (2 P)

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

defines *Heaviside's step function.*

Show that the derivative of the step function is a delta-function, i.e. $\langle \Theta', f \rangle = \langle \delta, f \rangle = f(0)$ gilt.

2 The Green function (10 P) The Fourier transform plays a tantamount role in the theory of inhomogeneous, linear differential equations. If – as was shown in the lecture – G is a so called *fundamental solution* of the differential equation

$$\mathcal{L}f = g$$

i.e.

$$\mathcal{L}G(x, x') = \delta(x - x')$$

one may calculate a particular solution for an inhomogeneity g by convolution

$$f(x) = (G * g)(x) = \int G(x, x')g(x')\mathrm{d}x$$

G is called *Green function.*
Since the Fourier transform maps derivatives to multiplications, it simplifies the calculation of the Green function to an algebraic problem and subsequent Fourier inversion. An example.

a) (2 P) Again, we look at the damped harmonic oscillator. Let

$$\mathcal{L}G(t, t') = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right) G(t, t') = \delta(t - t') \quad (1)$$

We are interested in the behavior of the oscillator under the influence of an arbitrary driving force[1] $g(t)$.

Map (1) by Fourier transformation to an algebraic equation and determine the Green function $G(t, t')$ up to execution of the inversion integral. We are going to explicitly solve this integral later with methods from complex calculus.

b) (4 P) Fourier inversion yields the Green function

$$G(t, t') = e^{-\gamma(t-t')}\frac{\sin(\omega(t - t'))}{\omega}\Theta(t - t') \quad (2)$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2}$ is the natural frequency of the oscillator and Θ is the Heaviside function that was introduced in problem 1.

Interpret this expression physically. Does it fit your understanding of Green function as a "pulse response" of the system? What does the step function imply with regards to causality?

c) (4 P) Show that (2) is indeed the proper Green function of the problem. I.e. show that for an arbitray test-function[2] ϕ

$$\int_{-\infty}^{\infty} \mathcal{L}G(t, t')\phi(t)\mathrm{d}t = \int \delta(t - t')\phi(t)\mathrm{d}t = \phi(t')$$

Hint: Use the properties from problem 1.

[1] Up to now we were restricted to sinusoidal forces. The formalism of the Green function finally enables us to treat general situations.

[2] Test-functions are particularly well-behaved. They are infinitely differentiable. Furthermore, a test-function as well as all its derivatives decays faster than any polynomial at infinity. A Gaussian is a good example.

3 Heat conduction in one dimension (13 P) Consider a metal rod which for practical reasons we assume as infinity long. The temperature at a position x at time t is recorded in a function $T(x,t)$. Heat flows wherever there's a temperature gradient. If the specific heat of the metal is assumed to be a constant, the temperature profile T fulfills the heat equation

$$\frac{\partial T}{\partial t} - D\frac{\partial^2 T}{\partial x^2} = 0 \tag{3}$$

D is the thermal diffusivity. The goal is to solve this equation by means of Fourier-transformation. At time $t = 0$ the temperature profile shall be given by a function $T(x,0)$.

a) (2 P) Fourier-transform (3) in x and solve the generated ODE in t. Call the introduced Fourier-variable k. The initial condition is a function $c(k)$. Deduce that

$$c(k) = \tilde{T}(k,0)$$

The function $c(k)$ is therefore the Fourier transform of the initial temperature profile $T(x,0)$.

b) (3 P) Write $T(x,t)$ as a Fourier inversion. Express $\tilde{T}(k,0)$ through $T(x,0)$.

Intermediate result: $T(x,t) = \frac{1}{2\pi}\int e^{-Dk^2t-ik(x-x')}T(x',0)\,\mathrm{d}k\,\mathrm{d}x'$

c) (5 P) Execute the integral over k and determine the through

$$T(x,t) = \int H(x-x';t)T(x',0)\mathrm{d}x'$$

defined *heat kernel* H.

d) (3 P) Draw for different times the fate of a temperature peak initially located at $x = 0$, $T(x,0) = \alpha\delta(x)$. You may set $D = 1$. What is the unit of α?