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Aufgrund von Christi-Himmelfahrt haben Sie bis Freitag, 31.05 Zeit zur Bearbeitung.
Die Übungsstunden am 30.05 entfallen. Wir versuchen stattdessen eine gemeinsame Stunde für Mittwoch den 29.05 zu organiseren. Beachten Sie bitte die Ankündigung auf der Webseite.

1 Fourier-Tranformation (6 P) Up until now we used the Fourier-transform as a tool to solve differential equations, but haven't actually calculated a transform. We rectify this now. Find the Fourier transform of the functions ($L > 0$)

a) (3 P)

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & -L < x < L \\ 0 & \text{else} \end{cases}$$

b) (3 P)

$$g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^{-|x|/L}$$

2 Complex differentiability (9 P)

a) (5 P) Decide at which points the following functions are complex differentiable.

$$\begin{aligned} f_1(z) &= z^4, \\ f_2(z) &= \mathrm{e}^z, \\ f_3(z) &= |z|^2, \\ f_4(z) &= \sin(z), \\ f_5(z) &= z^*. \end{aligned}$$

Where are they holomorphic?

Instead of the usual coordinates x, y along the basis vectors $\{1, \mathrm{i}\}$, we may consider different coordinate systems in the complex plane.
An interesting and sometimes helpful choice are the coordinates

$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y$$

b) (2 P) Show that

$$\frac{\partial f(x,y)}{\partial z} = \frac{1}{2}\left(\frac{\partial f(x,y)}{\partial x} - \mathrm{i}\frac{\partial f(x,y)}{\partial y}\right)$$

and

$$\frac{\partial f(x,y)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f(x,y)}{\partial x} + \mathrm{i}\frac{\partial f(x,y)}{\partial y}\right)$$

The differential operators $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) f$ und $\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) f$ are called *Wirtinger derivatives*. The second one in particular comes in handy, due to the following statement.

c) (2 P) Show that f fulfills the Cauchy-Riemann-condition (i.e. is holomorphic) if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

This provides an elegant way to proof complex differentiability without the need to split a function into real and imaginary part first.

3 Complex line integral (9 P)

Consider the paths

$$\gamma_1 : [0,1] \to \mathbb{C}, \quad \gamma_1(t) = (1+i)t$$
$$\gamma_2 : [0,1] \to \mathbb{C}, \quad \gamma_2(t) = t + it^2$$

a) (2 P) Draw γ_1 and calculate $\int_{\gamma_1} z^2 \mathrm{d}z$.

b) (2 P) Draw γ_2 and calculate $\int_{\gamma_2} z^2 \mathrm{d}z$. Do you notice anything peculiar?

c) (2 P) The *length* of a differentiable curve $\gamma : [t_0, t_1] \to \mathbb{C}$ is defined as the number

$$L(\gamma) := \int_{t_0}^{t_1} |\dot{\gamma}(t)| \, \mathrm{d}t.$$

Determine $L(\gamma_1)$ and $L(\gamma_2)$.

d) (3 P) Let $f : \Omega \to \mathbb{C}$ be continuous and bounded $|f(z)| \leq C \; \forall z \in \Omega$.

Further let $\gamma : [t_0, t_1] \to \Omega$ a continuously differentiable path. Proof the following upper bound:

$$\left| \int_\gamma f(z) \mathrm{d}z \right| \leq C L(\gamma).$$

4 Consequences of Cauchy's integral formula (6 P)

From the lecture:

Theorem 1 (Cauchy's integral formula on disks) *Let $f : \Omega \to \mathbb{C}$ holomorphic on an open set $\Omega \subseteq \mathbb{C}$ that contains the disk $\{z : \; |z - z_0| \leq r\}$. Then, for any $a \in \mathbb{C}$ in the interior of this disk*

$$f(a) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-a} \mathrm{d}z.$$

holds.

a) (3 P) Determine the following line integral using Cauchy's formula:

$$\int_{|z-1|=4} \frac{\mathrm{e}^{3z}}{z - \pi i} \mathrm{d}z.$$

b) (3 P) Determine

$$\int_0^{2\pi} \sin\left(r\mathrm{e}^{it}\right) \mathrm{d}t \quad \text{und} \quad \int_0^{2\pi} \cos\left(r\mathrm{e}^{it}\right) \mathrm{d}t$$

für $r = 1$ und $r = 10$. Does the value of r matter at all?