Lineare Algebra und Vektoranalysis
Sommersemester 2019
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1 More about complex differentiablity (6 P) Check that the following functions u are harmonic ($\Delta u = 0$). Find in both cases another harmonic function v, such that $u + \mathrm{i}v$ is analytic.

a) $u_1(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$

b) $u_2(x, y) = \ln\left(x^2 + y^2\right)$

2 A real integral (4 P) Determine the following real integral by converting it into a line integral along a suitable path.

$$\int_0^{2\pi} \frac{1}{2 + \cos t}\,\mathrm{d}t$$

3 Liouville's theorem (10 P) A function that is holomorphic on all of $\mathbb{C}$ is called *entire*. In particular, an entire function can be expanded into a power series with infinite radius of convergence around any point.
A function f is called *bounded* if there exists $0 \leq M < \infty$ such that for all z from the domain $|f(z)| \leq M$ holds.

a) (4 P) Let $D \subset \mathbb{C}$ be the disk of radius ρ and center a. It boundary is K. Let further f be holomorphic on $D \cup K$. Show the following bound:

$$\left|f^{(n)}(a)\right| \leq \frac{n! \max_{z\in K}\{|f(z)|\}}{\rho^n}$$

b) (2 P)Equipped with this, proof

Liouville's theorem: *A bounded, entire function* $f : \mathbb{C} \to \mathbb{C}$ *is constant.*

c) (1 P) Why isn't $f = \sin$ not in contradiction to the theorem?

d) (3 P) Using Liouville's theorem, proof the

Fundamental theorem of algebra: *Any non-constant polynomial p with complex coefficients has at least one complex root.*

Hint: Consider the function $1/p$ and proof the statement by contradiction.

Deduce from this, that a polynomial of degree n has exactly n complex roots.

4 Connection of Taylor- and Fourier-serieses (10 P) Let f be a holomorphic function on U, where U contains the.
Write $z = re^{\mathrm{i}\phi}$ inside the radius of convergence of the series expansion of f around zero.

a) (1 P) Show that $f(z) = \sum_n a_n r^n \cos n\phi + \mathrm{i} \sum_n a_n r^n \sin n\phi$.

If one splits the function $f = u + \mathrm{i}v$ into real and imaginary part, the complex power series yields Taylor series in r, as well as Fourier seris in ϕ.

b) (3 P) Determine the Fourier series of the real function

$$v(\phi) = \frac{2\sin\phi}{5 - 4\cos\phi}$$

c) (3 P) Show that for $m \in \mathbb{N}$

$$\int_0^{2\pi} v(\phi)\sin(m\phi)\,\mathrm{d}\phi = \frac{\pi}{2^m}$$

d) (3 P) Determine the Taylor expansion of the real function

$$\tilde{v}(r) = \frac{r}{\sqrt{2}(1 + r^2) - 2r}$$

around $r = 0$.

Hint for b)-d): Consider the power series of $(1 - z)^{-1}$.